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Théorie descriptive des ensembles et espaces de Banach

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Résumé

Résumé

Cette thèse traite de la théorie descriptive des ensembles et de la géométrie des espaces de Banach. La première partie consiste en l'étude de la complexité descriptive de la famille des espaces de Banach avec la propriété d'approximation bornée, respectivement la propriété π, dans l'ensemble des sous-espaces fermés de $C(\Delta)$, où Δ est l'ensemble de Cantor. Ces familles sont boréliennes. En outre, nous montrons que si $\alpha < \omega_1$, l'ensemble des espaces d'indice de Szlenk au plus α qui ont une FDD contractante est borélien. Nous montrons dans la seconde partie que le nombre de classes d'isomorphisme de sous-espaces complémentés des espaces d'Orlicz de fonctions réflexive $L^\Phi[0,1]$ est non dénombrable, où $L^\Phi[0,1]$ n'est pas isomorphe à $L^2[0,1]$.

Mots-clefs

Propriété d'approximation bornée, propriété π_λ, FDD, espace d'Orlicz de fonctions, groupe de Cantor, espace invariant par réarrangement.

Descriptive set theory and Banach spaces

Abstract

This thesis deals with the descriptive set theory and the geometry of Banach spaces. The first chapter consists of the study of the descriptive complexity of the set of Banach spaces with the Bounded Approximation Property, respectively π-property, in the set of all closed subspaces of $C(\Delta)$, where Δ is the Cantor set. We show that these sets are Borel. In addition, we show that if $\alpha < \omega_1$, the set of spaces with Szlenk index at most α which have a shrinking FDD is Borel. We show in the second chapter that the number of isomorphism classes of complemented subspaces of the reflexive Orlicz function space $L^\Phi[0,1]$ is uncountable, where $L^\Phi[0,1]$ is not isomorphic to $L^2[0,1]$.

Keywords

Bounded Approximation Property, π_λ-property, FDD, Orlicz function space, Cantor group, rearrangement invariant function space.

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Introduction

0.1 French version

Ce travail est consacré à la théorie descriptive des ensembles et aux espaces de Banach. Il est divisé en deux parties. Dans la premier partie, nous montrons que la famille des espaces de Banach avec la propriété d'approximation bornée (BAP), respectivement avec la propriété (π), est borelienne dans l'ensemble des sous-espaces fermés de $C(\Delta)$, où Δ est l'ensemble de Cantor, équipé de la structure borélienne d'Effros. Dans la deuxième partie, nous montrons que le nombre de classes d'isomorphisme de sous-espaces complémentés des espaces d'Orlicz de fonctions réflexive $L^{\Phi}[0,1]$ est non dénombrable, oú $L^{\Phi}[0,1]$ n'est pas isomorphe à $L^2[0,1]$. De plus, nous montrons que la classe d'isomorphisme d'un tel espace $L^{\Phi}[0,1]$ est analytique non borélienne. Enfin, en utilisant le théorème d'interpolation de Boyd nous étendons quelques résultats connus sur les espaces $L^p[0,1]$ aux espaces de fonctions invariants par réarrangement satisfaisant les conditions naturelles sur leurs indices de Boyd.

La complexité descriptive de la famille des espaces de Banach avec la propriété d'approximation bornée (BAP), respectivement avec la propriété (π)

Soit $C(\Delta)$ l'espace des fonctions continues sur l'ensemble de Cantor Δ. Il est bien connu que $C(\Delta)$ est isométriquement universel pour les espaces de Banach séparables. Nous notons $\mathcal{SE}$ l'ensemble des sous-espaces fermés de $C(\Delta)$ équipé de la structure borélienne d'Effros. Dans [Bos02], B. Bossard a considéré la complexité de la relation d'isomorphisme et celle de nombreux sous-ensembles de $\mathcal{SE}$. Nous rappelons qu'un espace de Banach X a la λ- propriété d'approximation bornée (λ-BAP), $\lambda \geqslant 1$, si pour tout compact $K \subset X$ et tout $\epsilon > 0$, il existe un opérateur de rang fini $T : X \longrightarrow X$ tel que $\|T\| \leqslant \lambda$ et $\|T(x) - x\| < \epsilon$ pour tout $x \in K$.

Nous montrons que le sous-ensemble de $\mathcal{SE}$ formé des espaces de Banach séparables qui ont la BAP est borélien. De plus, nous montrons que si X est un espace de Banach séparable avec une M-base normante $\{e_i, e_i^*\}_{i=1}^{\infty}$ et que $E_u = \overline{span}\{e_i;\ i \in u\}$ pour $u \in \Delta$, l'ensemble $\{u \in \Delta;\ E_u \textit{ a une FDD}\}$ est comaigre dans Δ. Nous rappelons qu'un espace X a la propriété (π_λ) s'il existe un filtre de projections (S_α) sur X qui converge fortement vers l'identité sur X avec $\limsup_\alpha \|S_\alpha\| \leqslant \lambda$, (voir [Cas01]). On dit que X a la propriété (π) s'il a la propriété (π_λ) pour un certain $\lambda \geq 1$. L'espace séparable X a une décomposition finie-dimensionnelle (une FDD) $\{X_n\}_{n=1}^{\infty}$, où les X_n sont des sous-espaces de dimension finie de X, si pour tout $x \in X$ il existe une unique suite (x_n) telle que $x_n \in X_n$, $n \in \mathbb{N}$ et $x = \sum_{n=1}^{\infty} x_n$. Un espace de Banach séparable a une FDD si et seulement si il a la propriété

(π) et la BAP avec une suite approximante (R_n) formée d'opérateurs de rang fini qui commutent (cette propriéte est notée CBAP).

Nous montrons que l'ensemble des espaces séparables qui ont la propriété (π) est borélien dans $\mathcal{SE}$. On en déduit quelques corollaires sur la complexité de familles d'espaces avec une décomposition finie-dimensionnelle (FDD). Par exemple, nous montrons que l'ensemble des espaces de Banach d'indice de Szlenk borné par un certain ordinal α et qui ont une FDD contractante est borélien.

Au début de la partie 1, nous rappelons quelques résultats de base de théorie descriptive, ainsi que des notions de géométrie des espaces de Banach, comme la Fréchet-différentiabilité et les indices de Szlenk.
Dans la section 1.4, nous montrons que le sous-ensemble de $\mathcal{SE}$ formé des espaces de Banach séparables qui ont la BAP est borélien. Ce résultat repose sur le lemme technique suivant :

Lemme 0.1.1. *Soit $(x_n)_{n=1}^{\infty}$ une suite dense d'un espace de Banach X et $\lambda \geqslant 1$. Alors X a la λ-BAP si et seulement si*

$$\forall K \ \forall \epsilon > 0 \ \exists \lambda' < \lambda \ \exists R \ \forall N \ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R \ \forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{0.1.1}$$

$$\forall i \leqslant K, \ \|x_i - [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \epsilon \tag{0.1.2}$$

où K, R, N varient dans $\mathbb{N}$ et ϵ, λ' parcourt $\mathbb{Q}$.

On montre alors :

Théorème 0.1.2. *L'ensemble des espaces de Banach séparables qui ont la BAP est borélien.*

De plus, la démonstration montre :

Proposition 0.1.3. *L'application $\psi : \{X \in \mathcal{SE}; \ X \text{ has the BAP}\} \longrightarrow [1, \infty[$ définie par $\psi(X) = inf\{\lambda; \ X \text{ a la } \lambda - BAP\}$, est borélienne.*

On ne sait pas si l'ensemble des espaces séparables qui ont une base est borélien ou non. S'il ne l'est pas, cela implique que l'ensemble des espaces de Banach séparables ayant la BAP mais sans base est conanalytique non borélien. S. J. Szarek ([Sza87]) a montré que cet ensemble n'est pas vide par une construction trés délicate.

Nous montrons dans la section 1.5 que la famille des espaces de Banach qui ont la propriété (π) est borélienne. Le lemme technique est cette fois :

Lemme 0.1.4. *Soit $(x_n)_{n=1}^{\infty}$ une suite dense dans un espace de Banach X. L'espace X a la propriété (π) si et seulement si :*
$\exists \lambda > 1 \ \forall c \in (0, \frac{1}{4}) \bigcap \mathbb{Q} \ \forall K \ \forall \epsilon > 0 \ \forall \lambda' > \lambda \ \exists R \ \forall N \geqslant R \ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R$ $\forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{0.1.3}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^{R} \sigma_i(j)x_j]\| \leqslant \epsilon \tag{0.1.4}$$

$$\|\sum_{i=1}^{N}\alpha_i[\sum_{j=1}^{R}\sigma_i(j)x_j] - \sum_{i=1}^{N}\alpha_i[\sum_{t=1}^{R}[\sum_{j=1}^{R}\sigma_i(j)\sigma_j(t)]x_t]\| \leqslant c\cdot\|\sum_{i=1}^{N}\alpha_i x_i\|. \tag{0.1.5}$$

où K, R, N sont dans $\mathbb{N}$ et ϵ, λ', et λ sont dans $\mathbb{Q}$.

Il s'ensuit alors :

Théorème 0.1.5. *La propriété (π) est borélienne.*

La section 1.6 est consacrée à quelques résultats sur les espaces à dual séparable. L'ensemble $\mathcal{SD}$ des espaces à dual séparable est coanalytique dans $\mathcal{SE}$ et l'indice de Szlenk Sz est un rang coanalytique sur $\mathcal{SD}$. En particulier, les ensembles $S_\alpha = \{X \in \mathcal{SE};\ Sz(X) \leqslant \alpha\}$ sont boréliens dans $\mathcal{SE}$. Dans ces ensembles, on montre à l'aide du théorème de séparation de Souslin le résultat suivant.

Théorème 0.1.6. *Le sous-ensemble de S_α formé des espaces qui ont une FDD contractante est borélien dans $\mathcal{SE}$.*

Questions : Un espace séparable a la CBAP si et seulement si il a une norme équivalente qui a la MAP. Il s'ensuit que l'ensemble $\{X \in \mathcal{SE}; X\ a\ la\ CBAP\}$ est analytique. On ne sait pas s'il est borélien. On ne sait pas non plus s'il existe un sous-ensemble borélien B de $\mathcal{SE}$ tel que $\{X \in \mathcal{SD};\ X^*\ a\ la\ BAP\} = B \cap \mathcal{SD}$. Ce serait un renforcement du théorème (0.1.6). Enfin, ce qui arrive quand on remplace FDD par base n'est pas clair : par exemple, le sous-ensemble de S_α formé des espaces qui ont une base est clairement analytique. Est-il borélien ?

Il est naturel d'envisager des applications du théorème de Baire aux sous-espaces blocs d'un espace de Banach muni d'une base ou d'un système de coordonnées (voir par exemple [FG12]). Rappelons qu'un système biorthogonal $\{e_i, e_i^*\}_{i=1}^{\infty}$ dans $X \times X^*$ est appelé une base de Markushevich (M-base) de $(X, \|.\|)$ s'il est total et fondamental. De plus, un système biorthogonal $\{e_i, e_i^*\}_{i=1}^{\infty}$ dans $X \times X^*$ est dit λ- normant si la fonctionnelle $|||x||| = sup\{|x^*(x)|;\ x^* \in B_{X^*} \bigcap \overline{span}^{\|\cdot\|}\{e_i^*\}_{i=1}^{\infty}\}$, $x \in X$, est une norme qui satisfait $\lambda||x|| \leqslant |||x|||$ pour un certain $0 < \lambda \leqslant 1$, (voir [HZSV07]). Tout espace de Banach séparable a une M-base bornée normante (voir [Ter94]).

Théorème 0.1.7. *Soit X un espace de Banach séparable muni d'une M-base bornée normante $\{e_i, e_i^*\}_{i=1}^{\infty}$. On pose $E_u = \overline{span}\{e_i;\ i \in u\}$, où $u \in \Delta$. Alors l'ensemble $\{u \in \Delta;\ E_u\ a\ une\ FDD\}$ est comaigre dans l'ensemble de Cantor Δ.*

Notons que comme tout espace ayant une FDD a la BAP, le théorème (0.1.7) montre que $\{u \in \Delta;\ E_u\ a\ la\ BAP\}$ est comaigre dans Δ. Un exemple d'espace de Banach avec la BAP mais sans FDD a été construit par C. Read (voir [CK91]).

Classes d'isomorphismes des sous-espaces complémentés des espaces d'Orlicz de fonctions réflexive $L^\Phi[0, 1]$

Soit $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. On considére le groupe de Cantor $G = \{-1, 1\}^{\mathcal{C}}$ muni de la mesure de Haar. Son groupe dual est le groupe discret des fonctions de Walsh $w_F = \prod_{c \in F} r_c$ où F

est un sous-ensemble fini de $\mathcal{C}$ et r_c est la fonction de Rademacher, telle que $r_c(x) = x(c)$, $x \in G$. Ces fonctions de Walsh engendrent $L^p(G)$ pour $1 \leq p < \infty$, ainsi que espace d'Orlicz de fonctions réflexive $L^\Phi(G)$.

Une fonction mesurable f dans G ne dépend que des coordonnées de $F \subset \mathcal{C}$, si $f(x) = f(y)$ dès que $x, y \in G$ satisfont $x(c) = y(c)$ pour tout $c \in F$. Un sous-ensemble mesurable S de G ne dépend que des coordonnées de $F \subset \mathcal{C}$ si c'est le cas de χ_S. Pour $F \subset \mathcal{C}$, la σ-algèbre des sous-ensembles mesurables qui ne dépendent que des coordonnées de F est notée $\mathfrak{G}(F)$. Une branche de $\mathcal{C}$ est un sous-ensemble de $\mathcal{C}$ constitué d'éléments comparables (voir [BRS81], [Bou81] et [DK14]).

Dans [BRS81], les auteurs considèrent le sous-espace $X^p_{\mathcal{C}}$ défini comme le sous-espace fermé de $L^p(G)$ engendré par toutes les fonctions qui ne dépendent que des coordonnées de Γ, où Γ est une branche arbitraire de $\mathcal{C}$. Ils montrent que $X^p_{\mathcal{C}}$ est complémenté dans $L^p(G)$ et isomorphe à L^p, pour $1 < p < \infty$. Si T est un arbre sur $\mathbb{N}$, l'espace X^p_T est le sous-espace fermé de $L^p(G)$ engendré par toutes les fonctions qui ne dépendent que des coordonnées de Γ, où Γ est une branche arbitraire de T. Alors, X^p_T est 1-complémenté dans $X^p_{\mathcal{C}}$ par l'opérateur d'espérance conditionnelle par rapport à la σ-algèbre $\mathfrak{G}(T)$ formée des sous-ensembles de G qui ne dépendent que des coordonnées de T. J. Bourgain ([Bou81]) a montré que l'arbre T est bien fondé si et seulement si l'espace X^p_T ne contient pas de sous-espace isomorphe à $L^p[0,1]$, pour $1 < p < \infty$ et $p \neq 2$. On en déduit que si B est un espace de Banach séparable universel pour les espaces $\{X^p_T;\ T \textit{ est un arbre bien fondé}\}$, alors B contient une copie de $L^p[0,1]$. Il s'ensuit qu'on peut trouver un sous-ensemble non dénombrable d'espaces non isomorphes deux à deux dans cette classe.

Dans le chapitre 2, nous montrons que ces résultats s'étendent aux espaces d'Orlicz de fonctions réflexive $L^\Phi[0,1]$. De plus, certains résultats s'étendent aux espaces de fonctions invariants par réarrangement, sous certaines conditions sur les indices de Boyd.

Dans la section 2.1 nous donnons la définition d'un espace invariant par réarrangement X et de ses indices de Boyd α_X, β_X. Nous rappelons dans la section 2.2 le théorème d'interpolation de Boyd et sa version faible (voir [Boy69], [LT79] et [JMST79, section 8]). Les espaces d'Orlicz sont un cas particulier d'espaces invariants par réarrangement. Nous rappelons quelques résultats préliminaires sur les espaces d'Orlicz dont nous aurons besoin.

Nous utiliserons également des résultats importants de N. J. Kalton sur la propriété isométrique (M), ([Kal93]), à savoir :

Proposition 0.1.8. ***[Kal93, Proposition 4.1].*** *Un espace modulaire de suites $X = h_{(\Phi_k)}$ a une norme équivalente avec la propriété (M).*

Théorème 0.1.9. ***[Kal93, Théorème 4.3].*** *Soit X un espace de Banach séparable réticulé, sans atome et continu pour l'ordre. Si X a une norme équivalente avec (M), alors X est isomorphe $L^2[0,1]$ pour la norme et pour l'ordre.*

Ce théorème signifie que, mis à part l'espace de Hilbert, l'existence d'une norme avec la propriété (M) sur un réticulé l'oblige à être atomique.
En lien avec l'espace de Rosenthal X_p ($2 \leqslant p < \infty$). W.B. Johnson, B. Maurey, G. Schechtman et L. Tzafriri ([JMST79]) ont montré qu'à tout espace invariant par réarrangement Y sur $[0, \infty)$, on peut associer un espace U_Y avec une base inconditionnelle. Nous donnons quelques résultats sur ces espaces, dans le cas particulier des espaces d'Orlicz. Ensuite,

nous formulons quelques inégalités de martingales fondamentales dans le cadre des espaces d'Orlicz. Dans la section 2.7, nous étendons [BRS81, Theorem 1.1] aux espaces invariants par réarrangement $X[0,1]$ dont les indices de Boyd satisfont $0 < \beta_X \leq \alpha_X < 1$. Notre démonstration s'inspire de la preuve de [BRS81]. Nous utilisons des arguments d'interpolation pour la généraliser.

Théorème 0.1.10. *Soit $X[0,1]$ un espace invariant par réarrangement tel que $0 < \beta_X \leq \alpha_X < 1$. On suppose que $X[0,1]$ est isomorphe à un sous-espace complémenté d'un espace de Banach Y avec une décomposition de Schauder inconditionnelle (Y_j). Alors l'une au moins des assertions suivantes est vérifiée :*

(1) il existe i tel que $X[0,1]$ est isomorphe à un sous-espace complémenté de Y_i.

(2) Une suite bloc-base des Y_i's est équivalente à la base de Haar de $X[0,1]$ et engendre un sous-espace fermé complementé dans Y.

Dans la dernière partie du chapitre 2, nous généralisons [Bou81, Theorem (4.30)] aux espaces d'Orlicz de fonctions. Nous utilisons en particulier le théorème d'interpolation de Boyd et le théorème de Kalton ([Kal93], voir 0.2.9).

Soit donc $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. L'espace $X(G)$ est un espace invariant par réarrangement défini sur le groupe $G = \{-1,1\}^{\mathcal{C}}$ muni de la mesure de Haar. Les fonctions de Walsh w_F, où F est un sous-ensemble fini de $\mathcal{C}$, engendrent les espaces $L^p(G)$ pour tout $1 \leq p < \infty$, ainsi que les espaces $X(G)$.

Soit $X[0,1]$ un espace invariant par réarrangement tel que $0 < \beta_X \leqslant \alpha_X < 1$. Le sous-espace $X_{\mathcal{C}}$ est le sous-espace fermé de $X(G)$ engendré par toutes les fonctions qui ne dépendent que des coordonnées de Γ, où Γ est une branche arbitraire de $\mathcal{C}$. Donc $X_{\mathcal{C}}$ est le sous-espace de $X(G)$ engendré par les fonctions de Walsh

$$\{w_\Gamma = \prod_{c \in \Gamma} r_c;\ \Gamma \textit{ est une branche finie de } \mathcal{C}\}.$$

Si $T \subset \mathcal{C}$ est un arbre, nous définissons comme ci-dessus l'espace X_T. Nous utiliserons la notation X_T^Φ quand $X = L^\Phi(G)$est un espace d'Orlicz.

Proposition 0.1.11. *Soit $X[0,1]$ un espace invariant par réarrangement tel que $0 < \beta_X \leqslant \alpha_X < 1$. Alors $X_{\mathcal{C}}$ est un sous-espace complémenté de $X(G)$.*

Le résultat suivant est un corollaire de cette proposition. Cependant, nous en donnons également une preuve directe en utilisant les martingales, qui suit les étapes de la preuve de Bourgain.

Théorème 0.1.12. *Soit $L^\Phi(G)$ un espace d'Orlicz de fonctions. Alors $X_{\mathcal{C}}^\Phi$ est complémenté dans $L^\Phi(G)$.*

Notre résultat principal s'énonce maintenant comme suit.

Théorème 0.1.13. *Soit $L^\Phi[0,1]$ un espace d'Orlicz de fonctions réflexive qui n'est pas isomorphe à $L^2[0,1]$. Alors $L^\Phi[0,1]$ n'est pas isomorphe à un sous-espace de X_T^Φ si et seulement si T est un arbre bien fondé.*

Soit $\mathcal{T}$ l'ensemble des arbres sur $\mathbb{N}$. C'est un sous-ensemble fermé de l'ensemble de Cantor $\Delta = 2^{\mathbb{N}^{<\mathbb{N}}}$. Nous notons toujours $\mathcal{SE}$ l'ensemble des sous-espaces fermés de $C(\Delta)$ équipé de la structure borélienne d'Effros. On renvoie à [Bos02] et [AGR03] pour des applications de la théorie descriptive des ensembles aux espaces de Banach.

Lemme 0.1.14. *Soit $\psi : \mathcal{T} \mapsto \mathcal{SE}$ une application borélienne telle que si T est bien fondé, S est un arbre et $\psi(T) \cong \psi(S)$, alors S est bien fondé. Alors il existe une famille non dénombrable d'espaces deux à deux non isomorphes dans l'ensemble $\{\psi(T); T \text{ est bien fondé}\}$.*

Nous avons :

Lemme 0.1.15. *L'application $\psi : \mathcal{T} \mapsto \mathcal{SE}$ définie par $\psi(T) = X_T^\Phi$ est borélienne.*

On en déduit donc :

Corollaire 0.1.16. *Soit $L^\Phi[0,1]$ un espace d'Orlicz de fonctions réflexive qui n'est pas isomorphe à $L^2[0,1]$. Alors il existe une famille non dénombrable de sous-espaces complémentés de $L^\Phi[0,1]$ deux à deux non isomorphes.*

Un autre corollaire naturel est :

Corollaire 0.1.17. *Soit $L^\Phi[0,1]$ un espace d'Orlicz de fonctions réflexive qui n'est pas isomorphe à $L^2[0,1]$. Alors la classe d'isomorphisme $< L^\Phi[0,1] >$ de $L^\Phi[0,1]$ est analytique non borélienne.*

Il est montré dans [Bos02] que la classe d'isomorphisme $< \ell^2 >$ est borélienne. On ne sait pas si cette propriété caractérise l'espace de Hilbert, et nous rappelons donc [Bos02, Problem 2.9] : soit X un espace de Banach séparable dont la classe d'isomorphisme $< X >$ est borélienne. L'espace X est-il isomorphe à ℓ^2 ? Un cas particulier de cette question est particulièrement important : la classe d'isomorphisme $< c_0 >$ de c_0 est-elle borélienne ? Pour plus d'informations sur cette question et sur les familles analytiques d'espaces de Banach nous nous rapportons à [God10].

Appendice

Nous incluons les trois articles où les résultats de l'auteur sont publiés.

0.2 English version

This work deals with descriptive set theory and Banach spaces. It is divided into two parts. We mainly prove in the first part that the family of Banach spaces with the bounded approximation property (BAP), respectively with the π-property is a Borel subset of the set of all closed subspaces of $C(\Delta)$, where Δ is the Cantor set, equipped with the standard Effros-Borel structure. In the second part we show that the number of isomorphism classes of complemented subspaces of a reflexive Orlicz function space $L^\Phi[0,1]$ is uncountable, as soon as $L^\Phi[0,1]$ is not isomorphic to $L^2[0,1]$. Also, we prove that the set of all separable Banach spaces that are isomorphic to such an $L^\Phi[0,1]$ is analytic non Borel. Moreover, by using Boyd interpolation theorem we extend some results on $L^p[0,1]$ spaces to the rearrangement invariant function spaces under natural conditions on their Boyd indices.

The Descriptive Complexity of the Family of Banach Spaces with the Bounded Approximation Property and π-property

Let $C(\Delta)$ be the space of continuous functions on the Cantor space Δ. It is well-known that $C(\Delta)$ is isometrically isomorphic universal for all separable Banach spaces. We denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure. In [Bos02], B. Bossard considered the topological complexity of the isomorphism relation and many other subsets of $\mathcal{SE}$. We recall that the Banach space X has the λ- Bounded Approximation Property (λ-BAP), $\lambda \geqslant 1$, if for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \longrightarrow X$ with $\|T\| \leqslant \lambda$ and $\|T(x) - x\| < \epsilon$ for every $x \in K$.

We first show that the set of all separable Banach spaces that have the BAP is a Borel subset of $\mathcal{SE}$. Furthermore, we show that if X is a separable Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^{\infty}$ and $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u \textit{ has a FDD}\}$ is comeager in Δ. We recall that the Banach space X has the $\pi_\lambda-$property if there is a net of finite rank projections (S_α) on X converging strongly to the identity on X with $\limsup_\alpha \|S_\alpha\| \leqslant \lambda$, (see [Cas01]). We say that the Banach space X has the $\pi-$property if it has the π_λ-property for some $\lambda \geq 1$. A separable space X has a finite-dimensional decomposition (FDD) $\{X_n\}_{n=1}^{\infty}$, where each X_n is a finite-dimensional subspace of X, if for every $x \in X$ there exists a unique sequence (x_n) such that $x_n \in X_n$, $n \in \mathbb{N}$ and $x = \sum\limits_{n=1}^{\infty} x_n$. A separable Banach space has a FDD if and only if it has the π-property and BAP with an approximating sequence (R_n) consisting of commuting finite rank operators (this property is denoted CBAP).

We show that the set of all separable Banach spaces that have the $\pi-$property is a Borel subset of $\mathcal{SE}$. This bears some consequences on the complexity of the class of spaces with finite dimensional decompositions. For instance, we show that in the set of spaces whose Szlenk index is bounded by some countable ordinal, the subset consisting of spaces which have a shrinking finite dimensional decomposition is Borel.

Through the beginning of section 1, we present some preliminaries results in the descriptive set theory and Banach spaces such as the Fréchet derivative and Szlenk indices. In section 1.4, we prove the main result that the family of Banach spaces with the bounded approximation property is a Borel subset of the set $\mathcal{SE}$. We need the following technical lemma:

Lemma 0.2.1. *Suppose $(x_n)_{n=1}^{\infty}$ is a dense sequence in a Banach space X and $\lambda \geqslant 1$. Then X has the λ-bounded approximation property if and only if*

$$\forall K\ \forall \epsilon > 0\ \exists \lambda' < \lambda\ \exists R\ \forall N\ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R\ \forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{0.2.1}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \epsilon \tag{0.2.2}$$

where K, R, N vary over $\mathbb{N}$ and ϵ, λ' vary over $\mathbb{Q}$.

Then we can prove the following result:

Theorem 0.2.2. *The set of all separable Banach spaces that have the BAP is a Borel subset of $\mathcal{SE}$.*

Also, our argument shows:

Proposition 0.2.3. *The map* $\psi : \{X \in \mathcal{SE};\ X \text{ has the } BAP\} \longrightarrow [1, \infty[$, *defined by* $\psi(X) = inf\{\lambda;\ X \text{ has the } \lambda - BAP\}$, *is Borel.*

It is unknown if the set of all separable Banach spaces with a basis is Borel or not. If it is non Borel, then this will imply the non trivial result that the set of all separable Banach spaces with the BAP and without a basis is co-analytic non-Borel. It has been shown by S. J. Szarek in [Sza87] that this set is not empty.

We prove in section 1.5 that the family of Banach spaces with the π-property is a Borel subset of the set $\mathcal{SE}$. First we need the following result:

Lemma 0.2.4. *Suppose* $(x_n)_{n=1}^{\infty}$ *is a dense sequence in a Banach space* X. *Then* X *has the* π*-property if and only if*
$\exists\lambda > 1\ \forall c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}\ \forall K\ \forall\epsilon > 0\ \forall\lambda' > \lambda\ \exists R\ \forall N \geqslant R\ \exists\sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R$ $\forall\alpha_1, \cdots, \alpha_N \in \mathbb{Q}$

$$\|\sum_{i=1}^{N}\alpha_i[\sum_{j=1}^{R}\sigma_i(j)x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N}\alpha_i x_i\|. \tag{0.2.3}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^{R}\sigma_i(j)x_j]\| \leqslant \epsilon \tag{0.2.4}$$

$$\|\sum_{i=1}^{N}\alpha_i[\sum_{j=1}^{R}\sigma_i(j)x_j] - \sum_{i=1}^{N}\alpha_i[\sum_{t=1}^{R}[\sum_{j=1}^{R}\sigma_i(j)\sigma_j(t)]x_t]\| \leqslant c \cdot \|\sum_{i=1}^{N}\alpha_i x_i\|. \tag{0.2.5}$$

where $K,\ R,\ N$ *vary over* $\mathbb{N}$ *and* $\epsilon,\ \lambda'$, *and* λ *vary over* $\mathbb{Q}$.

Therefore, we have the main result:

Theorem 0.2.5. *The set of all separable Banach spaces that have the* π*-property is a Borel subset of* $\mathcal{SE}$.

In section 1.6 we prove results for the Banach spaces with separable dual. The set $\mathcal{SD}$ of all separable Banach spaces with separable dual spaces is co-analytic in $\mathcal{SE}$ and the Szlenk index Sz is a coanalytic rank on $\mathcal{SD}$. In particular, the set $S_\alpha = \{X \in \mathcal{SE};\ Sz(X) \leqslant \alpha\}$ is Borel in $\mathcal{SE}$. In this Borel set, the following holds.

Theorem 0.2.6. *The set of all separable Banach spaces in* S_α *that have a shrinking FDD is Borel in* $\mathcal{SE}$.

Questions: A separable Banach space has CBAP if and only if it has an equivalent norm for which it has MAP. It follows that the set $\{X \in \mathcal{SE}; X \text{ has the } CBAP\}$ is analytic. It is not clear if it is Borel or not. Also, it is not known if there is a Borel subset B of $\mathcal{SE}$ such that $\{X \in \mathcal{SD};\ X^* \text{ has the } AP\} = B \cap \mathcal{SD}$. This would be an improvement of Theorem (0.2.6). Finally, what happens when we replace FDD by basis is not clear: for instance, the set of all spaces in S_α which have a basis is clearly analytic. Is it Borel?

Some works have been done on the relation between Baire Category and families of subspaces of a Banach space with a Schauder basis, (see e.g. [FG12]). In order to state the next theorem we recall that a fundamental and total biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called a Markushevich basis (M-basis) for $(X, \|.\|)$. Furthermore, a biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called λ- norming if $|||x||| = sup\{|x^*(x)|;\ x^* \in$

$B_{X^*} \bigcap \overline{span}^{||\cdot||}\{e_i^*\}_{i=1}^\infty\}$, $x \in X$, is a norm satisfying $\lambda||x|| \leqslant |||x|||$ for some $0 < \lambda \leqslant 1$, (see [HZSV07]). Also, any separable Banach space has a bounded norming M-basis, (see [Ter94]). We recall that a separable Banach space X has a finite dimensional decomposition (a FDD) $\{X_n\}_{n=1}^\infty$, where X_n's are finite dimensional subspaces of X, if for every $x \in X$ there exists a unique sequence (x_n) with $x_n \in X_n$, $n \in \mathbb{N}$, such that $x = \sum\limits_{n=1}^{\infty} x_n$.

Theorem 0.2.7. *Let X be a separable Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^\infty$. If we let $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u$ has a FDD$\}$ is comeager in the Cantor space Δ.*

Note that since every space with a FDD has the BAP, Theorem (0.2.7) shows that $\{u \in \Delta;\ E_n$ *has the BAP*$\}$ is comeager in Δ. We recall that an example of a separable Banach space with the BAP but without a FDD has been constructed by C. Read, (see [CK91]).

Non-isomorphic Complemented Subspaces of Reflexive Orlicz Function Spaces $L^\Phi[0,1]$

Let $\mathcal{C} = \bigcup_{n=1}^\infty \mathbb{N}^n$. Consider the Cantor group $G = \{-1,1\}^{\mathcal{C}}$ equipped with the Haar measure. The dual group is the discrete group formed by Walsh functions $w_F = \prod_{c \in F} r_c$ where F is a finite subset of $\mathcal{C}$ and r_c is the Rademacher function, that is $r_c(x) = x(c)$, $x \in G$. These Walsh functions generate $L^p(G)$ for $1 \leq p < \infty$, and the reflexive Orlicz function spaces $L^\Phi(G)$.

A measurable function f on G only depends on the coordinates $F \subset \mathcal{C}$, provided $f(x) = f(y)$ whenever $x, y \in G$ with $x(c) = y(c)$ for all $c \in F$. A measurable subset S of G depends only on the coordinates $F \subset \mathcal{C}$ provided χ_S does. Moreover, For $F \subset \mathcal{C}$ the sub-σ-algebra $\mathfrak{G}(F)$ contains all measurable subsets of G that depend only on the F-coordinates. A branch in $\mathcal{C}$ will be a subset of $\mathcal{C}$ consisting of mutually comparable elements. For more the reader is referred to [BRS81], [Bou81] and [DK14].

In [BRS81], the authors considered the subspace $X_{\mathcal{C}}^p$ which is the closed linear span in $L^p(G)$ over all finite branches Γ in $\mathcal{C}$ of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. In addition, they proved that $X_{\mathcal{C}}^p$ is complemented in $L^p(G)$ and isomorphic to L^p, for $1 < p < \infty$. Moreover, for a tree T on $\mathbb{N}$, the space X_T^p is the closed linear span in $L^p(G)$ over all finite branches Γ in T of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. Hence, X_T^p is a one-complemented subspace of $X_{\mathcal{C}}^p$ by the conditional expectation operator with respect to the sub-σ-algebra $\mathfrak{G}(T)$ which contains all measurable subsets of G that depend only on the T-coordinates. J. Bourgain in [Bou81] showed that the tree T is well founded if and only if the space X_T^p does not contain a copy of $L^p[0,1]$, for $1 < p < \infty$ and $p \neq 2$. Consequently, it was shown that if B is a universal separable Banach space for the elements of the class $\{X_T^p;\ T$ *is a well founded tree*$\}$, then B contains a copy of $L^p[0,1]$. It follows that there are uncountably many mutually non-isomorphic members in this class.

In Chapter 2, we will show that these results extend to the case of the reflexive Orlicz function spaces $L^\Phi[0,1]$. Moreover, some of the results extend to rearrangement invariant function spaces (r.i. function spaces in short) under some conditions on the Boyd indices.

In section 2.1 we present the definition of the rearrangement invariant function space X and its Boyd indices α_X, β_X. Next, in section 2.2 we introduce the Boyd interpolation theorem and its weaker version, (see [Boy69], [LT79] and [JMST79, section 8]). In addition, as a special case of the rearrangement invariant function space we deal with the Orlicz spaces, so we give some preliminaries on Orlicz spaces that we need in our main results.

We need two important results of N. J. Kalton about the isometric property (M), (see [Kal93]), namely:

Proposition 0.2.8. ***[Kal93, Proposition 4.1].*** *A modular sequence space $X = h_{(\Phi_k)}$ can be equivalently normed to have property (M).*

Theorem 0.2.9. ***[Kal93, Theorem 4.3].*** *Let X be a separable order-continuous nonatomic Banach lattice. If X has an equivalent norm with property (M), then X is lattice-isomorphic to L^2.*

This theorem really says that, except for the Hilbert spaces, the existence of a norm with property (M) forces the lattice to be "discrete".
In connection with Rosenthal's spaces X_p for $2 \leqslant p < \infty$. W.B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in [JMST79] showed that to every r.i. function space Y on $[0, \infty)$ is associated a space U_Y with unconditional basis. We introduce some results about these spaces, especially about the Orlicz function spaces. After that, we state some basic probabilistic facts about the martingales inequalities for Orlicz function spaces. In section 2.7, we aim to extend [BRS81, Theorem 1.1] to the r.i. function space $X[0, 1]$ with the Boyd indices $0 < \beta_X \leq \alpha_X < 1$. Our proof heavily relies on the proof of [BRS81]. We will use interpolation arguments to extend it.

Theorem 0.2.10. *Let $X[0, 1]$ be a r.i. function space whose Boyd indices satisfy $0 < \beta_X \leq \alpha_X < 1$. Suppose $X[0, 1]$ is isomorphic to a complemented subspace of a Banach space Y with an unconditional Schauder decomposition (Y_j). Then one of the following holds:*

(1) There is an i so that $X[0, 1]$ is isomorphic to a complemented subspace of Y_i.

(2) A block basic sequence of the Y_i's is equivalent to the Haar basis of $X[0, 1]$ and has closed linear span complemented in Y.

In last section of Chapter 2, we will extend [Bou81, Theorem (4.30)] to the Orlicz function spaces. We will use in particular the Boyd interpolation theorem and Kalton's result [Kal93], see Theorem (0.2.9).

Again we let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. The space $X(G)$ is a r.i. function space defined on the separable measure space consisting of the Cantor group $G = \{-1, 1\}^{\mathcal{C}}$ equipped with the Haar measure. The Walsh functions w_F where F is a finite subset of $\mathcal{C}$ generate the $L^p(G)$ spaces for all $1 \leq p < \infty$. Then they also generate the r.i. function space $X(G)$.

We consider a r.i. function space $X[0, 1]$ such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. The subspace $X_{\mathcal{C}}$ is the closed linear span in the r.i. function space $X(G)$ over all finite branches Γ in $\mathcal{C}$ of the functions which depend only on the Γ-coordinates. Thus $X_{\mathcal{C}}$ is a subspace of $X(G)$ generated by Walsh functions $\{w_\Gamma = \prod_{c \in \Gamma} r_c;\ \Gamma$ *is a finite branch of* $\mathcal{C}\}$. If $T \subset \mathcal{C}$ is a tree, we define as above the space X_T. We will use the notation X_T^Φ when $X = L^\Phi(G)$ is an Orlicz space.

Proposition 0.2.11. *Let $X[0,1]$ be a r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Then $X_{\mathcal{C}}$ is a complemented subspace in $X(G)$.*

The following result directly follows from the previous proposition. However, we provide another proof by using the martingale sequences and following the steps of Bourgain's proof.

Theorem 0.2.12. *Let $L^\Phi(G)$ be a reflexive Orlicz function space. Then $X^\Phi_{\mathcal{C}}$ is complemented in $L^\Phi(G)$.*

Our main result reads now:

Theorem 0.2.13. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $L^\Phi[0,1]$ does not embed in X^Φ_T if and only if T is a well founded tree.*

Let $\mathcal{T}$ be the set of all trees on $\mathbb{N}$ which is a closed subset of the Cantor space $\Delta = 2^{\mathbb{N}^{<\mathbb{N}}}$. In addition, we denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure. For more about the application of descriptive set theory in the geometry of Banach spaces see e.g., [Bos02], or [AGR03].

Lemma 0.2.14. *Suppose $\psi : \mathcal{T} \mapsto \mathcal{SE}$ is a Borel map, such that if T is a well founded tree and S is a tree and $\psi(T) \cong \psi(S)$, then the tree S is well founded. Then there are uncountably many mutually non-isomorphic members in the class $\{\psi(T); T \text{ is well founded tree}\}$.*

We have:

Lemma 0.2.15. *The map $\psi : \mathcal{T} \mapsto \mathcal{SE}$ defined by $\psi(T) = X^\Phi_T$ is Borel.*

Therefore, we get:

Corollary 0.2.16. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0,1]$.*

Corollary 0.2.17. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then the isomorphism class $< L^\Phi >$ of L^Φ is analytic non Borel.*

In [Bos02], it has been shown that $< \ell^2 >$ is Borel. It is unknown whether this condition characterizes the Hilbert space, and thus we recall [Bos02, Problem 2.9]: Let X be a separable Banach space whose isomorphism class $< X >$ is Borel. Is X isomorphic to ℓ^2? A special case of this problem seems to be of particular importance, namely: is the isomorphism class $< c_0 >$ of c_0 Borel? For more about this question and analytic sets of Banach spaces see [God10].

Appendix

We include the three articles where the author's results are published.

Chapter 1

The Descriptive Complexity of the Family of Banach Spaces with the Bounded Approximation Property and π-property

In this chapter we will study the descriptive complexity of the family of Banach spaces with the bounded approximation property and the family of Banach spaces with the π-property. Based on finding approximate structures that are equivalent to the bounded approximation property and the π-property in the separable Banach space X, we prove that each family is Borel in $\mathcal{SE}$ where $\mathcal{SE}$ is the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure and $C(\Delta)$ is the space of continuous functions on the Cantor space Δ.

1.1 Preliminaries on the descriptive set theory

In this Chapter we mention definitions and results that we need in the next chapters about the descriptive set theory and its application in the geometry of Banach spaces depending on the references [Bos02], [Kec95] and [AGR03].

We denote $\omega = \{0, 1, 2, \cdots\}$ the set of natural numbers. The space ω^ω of sequences of integers is a metric complete separable space (i.e., a Polish space) when equipped with the metric

$$d(\sigma, \sigma') = \sum_{i \in \omega} 2^{-i} \delta(\sigma(i), \sigma'(i))$$

where $\delta(n, k) = 0$ if $n = k$ and 1 if not. It is easy to check that given any Polish space P, there exists a continuous onto map $S : \omega^\omega \to P$.

The following definition is due to Souslin and goes back to 1917.

Definition 1.1.1. A metric space M is *analytic* if there exists a continuous onto map $S : \omega^\omega \to M$.

Let P be a Polish space, that is, a separable completely metrizable space. A set $C \subseteq P$ is called *analytic* if there is a Polish space Y and a continuous function $f : Y \mapsto P$ with $f(Y) = C$. Moreover, if C is analytic, then $P \setminus C$ is *co-analytic*.

It follows immediately from the definition that the class of analytic sets is stable under continuous images. If A and B are sets, we define $\pi_1 : A \times B \to A$ by $\pi_1(a, b) = a$, and $\pi_2 : A \times B \to B$ by $\pi_2(a, b) = b$.

Lemma 1.1.2. *Let P be a Polish space, and $A \subseteq P$ be a subset of P. The following are equivalent:*

(i) A is analytic.

(ii) There exists a closed subset $F \subseteq P \times \omega^\omega$ such that $A = \pi_1(F)$.

Since $\omega^\omega \times \omega \simeq \omega^\omega$ and $(\omega^\omega)^\omega \simeq \omega^\omega$, this leads to:

Lemma 1.1.3. *The class of analytic subsets of a given Polish space P is stable under countable unions and countable intersections.*

Since closed subsets of the Polish space P are clearly analytic, Lemma (1.1.3) implies that every Borel subset of P is analytic, and thus one has:

Lemma 1.1.4. *If P_1 and P_2 are Polish spaces and $B \subseteq P_1 \times P_2$ is a Borel subset of $P_1 \times P_2$ then $\pi_2(B)$ is analytic.*

Definition 1.1.5. Let P be a standard Borel space and C be a co-analytic subset of P. A co-analytic rank r on C is a map $r : P \to \omega_1 \cup \{\omega_1\}$ such that

1. $C = \{x \in P;\ r(x) < \omega_1\}$.
2. $\{(x, y) \in C \times P;\ r(x) \leqslant r(y)\}$ is co-analytic in P^2.
3. $\{(x, y) \in C \times P;\ r(x) < r(y)\}$ is co-analytic in P^2.

For every co-analytic set C, there exists such a co-analytic rank, and the classical result below states that this rank is quite canonically associated with C.

Theorem 1.1.6. *Let r be a co-analytic rank of the co-analytic set C. Then:*

(i) For every $\alpha < \omega_1$, the set $B_\alpha = \{x \in C;\ r(x) \leqslant \alpha\}$ is Borel.

(ii) If $A \subseteq C$ is analytic, there is $\alpha < \omega_1$ such that $A \subseteq B_\alpha$.

(iii) If r' is another co-analytic rank on C, there exists $\phi : \omega_1 \to \omega_1$ such that if $r(x) \leqslant \alpha$, then $r'(x) \leqslant \phi(\alpha)$.

1.1.1 Trees and analytic sets

Now, we collect some results about trees on Polish spaces: Let $\mathbb{N}$ be a set of natural numbers. Given $n \in \mathbb{N}$, the *power* $\mathbb{N}^n$ is the set of all sequence (also called *nodes*) $s = (s(0), \cdots, s(n-1))$ of length n of elements from $\mathbb{N}$. If $m < n$, we let $s|m = (s(0), \cdots, s(m)) \in A^m$. In this situation, we say that $t = s|m$ is an *initial segment* of s and that s *extends* t, writing $t \leqslant s$. Two nodes are *compatible* if one is an initial segment of the other.

Definition 1.1.7. Let $\omega^{<\omega} = \cup_{n=0}^{\infty} \mathbb{N}^n$. A tree T on $\mathbb{N}$ is a subset of $\omega^{<\omega}$ closed under initial segments. The relation $\leqslant$ defined above induces a partial ordering on T.

Whenever convenient, we add a unique minimal element (*root*) to T, defined formally for $n = 0$ and denoted by $\emptyset$, that is the sequence of length zero. A *branch* of T is a linearly ordered subset of T, that is not properly contained in another linearly ordered subset of T. A tree is *well founded* if there is no infinite branch.

Let $\mathcal{T} \subset 2^{(\omega^{<\omega})}$ be the set of all trees on integers (including the tree $\{\emptyset\}$). This is a closed subset of $2^{(\omega^{<\omega})}$, and thus a compact set. Let $WF \subset \mathcal{T}$ be the subset consisting of well founded trees.

Let P be a Polish space with the distance d. If $x \in P$ and $r > 0$, $B(x,r) = \{y \in Y;\ d(x,y) < r\}$.
Let $A \subseteq P$ be analytic set. Then Lemma (1.1.2) implies that there exists a closed subset $F \subseteq P \times \omega^\omega$ such that $A = \pi_1(F)$. For $x \in P$, we define the tree $T(x) \in \mathcal{T}$ by:
$T(x) = \{(u_1, \cdots, u_n) \in \omega^{<\omega};\ \forall B(y, 1/n) \ni x,\ \exists x_n \in B(y, 1/n)\ and\ \exists \alpha \in \omega^\omega,\ \alpha|n = (u_1, \cdots, u_n)\ with\ (x_n, \alpha) \in F\}$.

Proposition 1.1.8. *Let $x \in P$, then $x \in A$ if and only if $T(x)$ is not well founded.*

Proof. Let $x \in A$. Then there exists $\alpha \in \omega^\omega$ such that $(x, \alpha) \in F$. Therefore, $\alpha|n \in T(x)$ for all n, then $T(x)$ is not well founded tree. Conversely, let $x \in P$ such that $T(x)$ is not well founded. Then there exists $\alpha \in \omega^\omega$ such that $\alpha|n \in T(x)$ for all n. By the definition of $T(x)$, there exists a sequence $(x_n, \beta_n) \in F$ with $d(x, x_n) \leqslant \frac{2}{n}$ and $\beta_n|n = \alpha|n$. Since this sequence converges to (x, α) in $P \times \omega^\omega$, then $(x, \alpha) \in F$ because F is closed. Hence, $x \in A$. □

Proposition 1.1.9. *For every $(\alpha, n) \in \omega^\omega \times \mathbb{N}$,*

$$\{x \in P;\ \alpha|n \in T(x)\}\ is\ closed. \tag{1.1.1}$$

In particular, the map $x \to T(x)$ is Borel from P to $\mathcal{T}$.
A map $x \to T(x)$ which verifies (1.1.1) is called s.c.s.

Proof. Let $u \in \omega^{<\omega}$ with $|u| = n$. We want to show Proposition (1.1.9) by proving that the subset $V_u = \{x \in P;\ u \notin T(x)\}$ is open in P. Then:
$u \notin T(x) \Leftrightarrow \exists B(y, 1/n) \ni x;\ \forall x' \in B(y, 1/n)\ and\ \forall \alpha \in \omega^\omega,\ \alpha|n = u \Rightarrow (x', \alpha) \notin F.$
Then $V_u \supset B(y, 1/n)$ which is a neighbourhood of x. Therefore, the subset V_u is open.
The fact that $T : P \to \mathcal{T}$ is Borel is an immediate implication of of the fact that $(\{T \in \mathcal{T}; u \in T\})_{u \in \omega^{<\omega}}$ is a base of the topology on $\mathcal{T}$ □

Theorem 1.1.10. *The set of well founded trees $WF \subseteq \mathcal{T}$ is co-analytic but not analytic.*

Proof. For showing that WF is co-analytic, we observe that

$$\mathcal{T} \setminus WF = \{T \in \mathcal{T};\ \exists \alpha \in \omega^\omega,\ \forall n \in \mathbb{N},\ \alpha|n \in T\}.$$

This set is the projection on $\mathcal{T}$ of the set $\{(T, \alpha) \in \mathcal{T} \times \omega^\omega;\ \forall n \in \mathbb{N},\ \ \alpha|n \in T\}$ which is a closed subset of $\mathcal{T} \times \omega^\omega$. Hence, the set WF is analytic.
Now, suppose WF is analytic: By Proposition (1.1.9) and Proposition (1.1.8), there exists a map $f : \mathcal{T} \to \mathcal{T}$ which is s.c.s. such that

$$x \in WF \Leftrightarrow f(x) \notin WF. \tag{1.1.2}$$

Then define $g(x) = x \cap f(x)$ and

$$T = \bigcup_{x \in \mathcal{T}} g(x).$$

show that $T \in WF$. Otherwise there exists $\alpha \in \omega^\omega$ and a sequence $(x_n) \subset \mathcal{T}$ such that $\alpha|n \in g(x_n)$ for $n \in \mathbb{N}$. Since $\mathcal{T}$ is compact, one can suppose that the sequence converges to $x \in \mathcal{T}$.
Define $F_n = \{y \in \mathcal{T};\ \alpha|n \in y\}$ and $G_n = \{y \in \mathcal{T};\ \alpha|n \in f(y)\}$.

The sequence (F_n) is a decreasing sequence of closed sets and it is the same for the sequence (G_n) since f is s.c.s. In addition, it is necessarily that x, the limit of x_n, belongs to $(\cap F_n)\cap(\cap G_n)$, since $x_n \in F_n \cap G_n$. Therefore, neither x nor $f(x)$ belongs to WF, and this impossible. Hence, T is a well founded tree.
Let $\widetilde{T} = \{u \in \omega^{<\omega};\ u \in T \textit{ or } u = (v,p) \textit{ with } v \in T,\ \textit{and}\, p \in \mathbb{N}\}$. The tree $\widetilde{T}$ is well founded and $g(\widetilde{T}) \subset T \subset \widetilde{T}$: The fact that $f(\widetilde{T}) \cap \widetilde{T} \subset T$ implies that $f(\widetilde{T}) \subset \widetilde{T}$ and therefore $f(\widetilde{T}) \in WF$. This contradicts that $\widetilde{T} \in WF$. □

1.1.2 The ordinal index of a tree

For a well founded tree, we inductively define a transfinite sequence of trees (T^α) on a set A as follows:

$$T^0 = T,$$

$$T^{\alpha+1} = \{(x_1,\cdots,x_n);\ (x_1,\cdots,x_n,x) \in T^\alpha \ \textit{for some}\ x \in A\},$$

$$T^\alpha = \cap_{\beta<\alpha} T^\beta \ \ \textit{for a limit ordinal}\ \ \alpha.$$

Since T is well founded, (T^α) is a strictly decreasing sequence, and thus $T^\alpha = \emptyset$ for some ordinal α. We define the ordinal index $\circ[T] = \min\{\alpha;\ T^\alpha = \emptyset\}$. In addition, if $T \subseteq \omega^{<\omega}$ and $T \notin WF$, we let $\circ[T] = \omega_1$, where ω_1 is the first uncountable ordinal. We also define for $x \in A$

$$T_x = \{(x_1,\cdots,x_n);\ (x,x_1,\cdots,x_n) \in T\}. \tag{1.1.3}$$

For $s = (x_1,\cdots,x_n) \in A^n$, $t = (y_1,\cdots,y_m) \in A^m$, we define the *concatenation* $s\hat{\ }t = (x_1,\cdots,x_n,y_1,\cdots,y_m) \in A^{n+m}$. Given trees $T \subset A^{<\omega}$, $S \subset B^{<\omega}$, we say that $\rho : S \mapsto T$ is a *regular map* if it preserves the lengths of nodes and the partial tree ordering. If there exists an injective regular map from S into T, we say that S is *isomorphic to a subtree* of T. Then we have $\circ[S] \leqslant \circ[T]$. On the other hand, if there exists a surjective regular map from S onto T, then we have $\circ[S] \geqslant \circ[T]$.

Lemma 1.1.11. *Let T be a well founded tree on A then:*
(i) $(T_x)^\alpha = (T^\alpha)_x$ for every ordinal α.
(ii) $\circ[T] = \sup_{x\in A}(\circ[T_x]+1)$.

Proof. It is easily verified by induction on α that $(T_x)^\alpha = (T^\alpha)_x$.
If $x \in A$ is fixed and $\alpha < \circ[T_x]$, then $(T_x)^\alpha = (T^\alpha)_x \neq \emptyset$ and therefore $x \in T^{\alpha+1}$. Distinguishing the cases $\circ[T_x]$ is not a limit ordinal, $\circ[T_x]$ is a limit ordinal, we see that $x \in T^{\circ[T_x]}$ and hence $\circ[T] \geqslant \circ[T_x]+1$. So $\circ[T] \geqslant \sup_{x\in A}\circ[T_x]+1$.
Let conversely $\alpha = \sup_{x\in A}\circ[T_x]+1$. For all $x \in A$ we have that $(T_x)^{\circ[T_x]} = (T^{\circ[T_x]})_x = \emptyset$. But this means that no complexes in $T^\alpha \subset T^{\circ[T_x]+1}$ starts with x. Thus, $T^\alpha = \emptyset$ and $\circ[T] \leqslant \alpha$. □

The following combinatorial result on trees is an abstract version of Souslin's separation theorem.

Theorem 1.1.12. *Let $\mathcal{T} \subseteq 2^{(\omega^{<\omega})}$ be the set of all trees on integers. For any countable ordinal $\alpha < \omega_1$, the set*

$$B_\alpha = \{T \in \mathcal{T};\ \circ[T] < \alpha\}$$

is a Borel subset of $\mathcal{T}$. Moreover, if A is analytic subset of $WF \subset \mathcal{T}$ then

$$\sup_{T \in A} \circ[T] < \omega_1.$$

In order to prove Theorem (1.1.12) we need the following lemma:

Lemma 1.1.13. *The set $\{(T, S) \in \mathcal{T} \times \mathcal{T};\ \circ[T] \leqslant \circ[S]\}$ is analytic in $\mathcal{T} \times \mathcal{T}$.*

Proof. We define the coding as an application $f : \omega^{<\omega} \to \omega^{<\omega}$ which transfers the sequences with length n to sequences with length n and satisfying the condition that for all $u, v \in \omega^{<\omega}$, we have

$$u \leqslant v \Rightarrow f(u) \leqslant f(v).$$

The set of all codings can be written as a closed set of the Polish space $\sqcap_n (\mathbb{N}^n)^{\mathbb{N}^n}$.
Let $S, T \in \mathcal{T}$. We observe that: $\circ[S] \leqslant \circ[T]$ if and only if there exists a coding f such that $S \subset f(T)$. The set $\{(T, S) \in \mathcal{T} \times \mathcal{T};\ \circ[S] \leqslant \circ[T]\}$ is a projection on $\mathcal{T} \times \mathcal{T}$ of the closed set $\{(S, T, f);\ S \subseteq f(T)\}$. □

Now, we return to proof Theorem (1.1.12).
Assume that $\sup_{T \in A} \circ[T] = \omega_1$. We can write

$$WF = \{S \in \mathcal{T};\ \exists T \in A \ \ \circ[S] \leqslant \circ[T]\},$$

and WF will be the projection of of the set

$$\{(S, T);\ T \in A\} \cap \{(S, T);\ \ \circ[S] \leqslant \circ[T]\}$$

which is analytic by the previous Lemma. Then WF is analytic and this contradicts Theorem (1.1.10).
finally, we can use the transfinite induction on $\alpha < \omega_1$ to prove that the sets B_α are Borel.

1.1.3 Lusin-Sierpinski index

Let P be a Polish space and $B \subseteq P$ be a co-analytic subset. There exists a Borel map T from P to $\mathcal{T}$ such that $x \to T(x)$ satisfies

$$x \in B \Leftrightarrow T(x) \in WF$$

We define then a map $\Psi : P \to \{\alpha; \alpha \leqslant \omega_1\}$ by for all $x \in P$, $\Psi(x) = \circ[T(x)]$.
The properties of this index is the same of $\circ[.]$ on WF.

(a) If $A \subset B$ is analytic then $\{T(x);\ x \in A\}$ is analytic in WF, since the map $r : P \to \mathcal{T}$, where $r(x) = T(x)$, is Borel. Therefore, Theorem (1.1.12) implies

$$\sup_{x \in A} \Psi(x) < \omega_1.$$

(b) If $\alpha < \omega_1$, then $\{x;\ \Psi(x) < \alpha\} = r^{-1}(B_\alpha)$ is Borel.

Theorem 1.1.14. *Let P be a Polish space and A_1, A_2 analytic disjoint subsets of P. Then there exist two Borel disjoint subsets A'_1, A'_2 contain A_1 and A_2 respectively.*

Proof. Let $B_1 = P \setminus A_1$ and Ψ_1 be Lusin-Sierpinski index associated with B_1. Since A_2 is analutic and $A_2 \subset B_1$, then there exists a countable ordinal α such that $\Psi_1(x) < \alpha$ when $x \in A_2$. It is sufficient to take $A'_2 = \{\Psi_1 < \alpha\}$ and $A'_1 = P \setminus A'_2$. □

1.2 Descriptive set theory and the geometry of Banach spaces

Let P be a Polish space, and $\mathcal{O}$ be a basis of open subsets of P. We denote by $\mathcal{F}(P)$ the set of all closed subsets of P equipped with the Effros-Borel structure (i.e. the canonical Borel structure generated by the family $(\{F \in \mathcal{F}(P) : F \cap U \neq \emptyset\})_{U \in \mathcal{O}}$, (see [Kec95])). If in addition P is compact, the Effros-Borel structure is generated by the Hausdorff topology, thus by the family $(\{F \in \mathcal{F}(P) : F \subseteq U\})_{U \in \mathcal{O}}$.

The following is a basic fact about the Effros Borel space.

Theorem 1.2.1. ***[Kec95, Theorem (12.13): The selection theorem for $\mathcal{F}(X)$].*** *Let X be Polish. There is a sequence of Borel functions $d_n : \mathcal{F}(X) \mapsto X$, such that for nonempty $F \in \mathcal{F}(X)$, $\{d_n(X)\}$ is dense in F.*

A Banach space is universal for separable Banach spaces if it contains an isomorphic copy of every separable Banach space. All separable Banach spaces can be realized, up to isometry, as subspaces of $C(\Delta)$, where $\Delta = 2^\omega$ is the Cantor set. Denoting by $\mathcal{SE}(X)$ the set of all closed linear subspaces of the separable Banach space X and endowing $\mathcal{SE}(X)$ with the relative Effros-Borel structure, the set $\mathcal{SE}(C(\Delta)) = \mathcal{SE}$ becomes the standard Borel space of all separable Banach spaces, (see [Bos02], [AGR03], [God10]).

Next, we mention some results that were proved by B. Bossard in [Bos02]:

Lemma 1.2.2. ***[Bos02, Lemma 2.6].*** *Let P be Polish space and Z be a separable Banach space.*

(i) $\{(F, y);\ y \in F\}$ *is Borel in* $\mathcal{F}(P) \times P$*, and consequently* $\{(F, y);\ y \in F\}$ *is Borel in* $\mathcal{SE}(Z) \times Z$.

(ii) $\{(Y, (y_i));\ \overline{span}(y_i) = Y\}$ *is Borel in* $\mathcal{SE}(Z) \times Z^\omega$.

(iii) $\{((x_i), (y_i));\ (x_i)$ *is equivalent to* $(y_i)\}$ *is Borel in* $Z^\omega \times Z^\omega$.

(iv) $\{(F, G);\ G \subseteq F\}$ *is Borel in* $\mathcal{F}(P) \times \mathcal{F}(P)$*, and consequently:*
$\{(X, Y);\ X$ is isomorphic to a subspace of $Y\}$ *is Borel in* $\mathcal{SE}(Z) \times \mathcal{SE}(Z)$

Theorem 1.2.3. ***[Bos02, Theorem2.3]***

(i) The isomorphism relation $\cong$ is analytic non-Borel in $\mathcal{SE}^2$, that is, the set

$$I = \{(X, Y) \in \mathcal{SE}^2;\ X \cong Y\}$$

is analytic non-Borel. There exists a space $U \in \mathcal{SE}$ whose isomorphism class

$$\langle U \rangle = \{X \in \mathcal{SE};\ X \cong U\}$$

is analytic non-Borel.

(ii) The relation $\{(X, Y);\ X$ is isomorphic to a (complemented) subspace of $Y\}$ *is analytic non-Borel in $\mathcal{SE}^2$.*

Moreover, the following holds.

Corollary 1.2.4. ***[Bos02, Corollary 3.3-(ii)].*** *The family of separable Banach spaces with separable dual is co-analytic non Borel.*

Denote by $\mathcal{SD}$ the set of all $X \in \mathcal{SE}$ with separable dual which is co-analytic non Borel (see Corollary (1.2.4). P. Dodos was concerned about the question whether analyticity is preserved under duality, he found out that: if A is an analytic class of separable dual Banach spaces, then the set $A_* = \{X \in \mathcal{SE};\ \exists Y \in A \textit{ with } X^* \cong Y\}$ is not necessary analytic. For a counterexample he took $A = \{Y \in \mathcal{SE};\ Y \cong \ell^1\}$, then $A_* = \{X \in \mathcal{SE},\ X^* \cong \ell^1\}$ is not analytic, (see [Dod10]). Also, he proved the following results.

Theorem 1.2.5. ***[Dod10, Theorem 1].*** *Let A be an analytic class of separable Banach spaces with separable dual. Then the set $A^* = \{Y \in \mathcal{SE}; \exists X \in A \text{ with } X^* \cong Y\}$ is analytic.*

Proposition 1.2.6. ***[Dod10, Proposition 7].*** *Let A be an analytic class of separable dual spaces. Let also B be an analytic subset of $\mathcal{SD}$. Then the set $A_*(B) = \{X \in B; \exists Y \in A \text{ with } X^* \cong Y\}$ is analytic.*

1.2.1 Szlenk indices

Let P be a Polish space. Every map d from $\mathcal{F}(P)$ to $\mathcal{F}(P)$ such that $d(F) \subseteq F$ for any $F \in \mathcal{F}(P)$, and $d(F) \subseteq d(F')$ if $F \subseteq F'$, is called a derivation.
If d is a derivation, we associate to it an ordinal index σ_d defined as follows. Let $F \in \mathcal{F}(P)$. We set $F^{(0)} = F$, and inductively define, for an ordinal α,

$$F^{(\alpha+1)} = d(F^\alpha)$$

and

$$F^{(\beta)} = \bigcap_{\alpha<\beta} F^{(\alpha)} \text{ if } \beta \text{ is a limit ordinal.}$$

Since P is Polish, for some $\alpha < \omega_1$ we have $F^{(\alpha+1)} = F^{(\alpha)}$. We let $\sigma_d(F) = min\{\alpha; F^{(\alpha)} = \emptyset\}$ if such an ordinal exists, and ω_1 otherwise.
We refer the reader to [Szl68] and [Lan06] for more about the Szlenk indices and their applications. Let X be a separable Banach space and $\epsilon > 0$. We define two derivations on $\mathcal{F}(B_{X^*})$ by
$\delta(\epsilon) : F \to F_\epsilon^{[\prime]} = \{x^* \in F;\ \|.\| - diam(H \cap F) \geqslant \epsilon\ for\ all\ w^* - open\ half - spaces\ H \ni x^*\}$,
$d(\epsilon) : F \to F'_\epsilon = \{x^* \in F;\ \|.\| - diam(V \cap F) \geqslant \epsilon\ for\ all\ w^* - open\ sets\ V \ni x^*\}$,
that is to say, F'_ϵ (resp. $F_\epsilon^{[\prime]}$) is what is left from F when all w^*-open subsets (resp. w^*-open slices) of diameter less than ϵ are removed.
We set $\zeta_\epsilon = \sigma_{d(\epsilon)}$, $\xi_\epsilon = \sigma_{\delta(\epsilon)}$ and

$$\zeta(F) = \sup_{\epsilon>0} \zeta_\epsilon(F), \quad \xi(F) = \sup_{\epsilon>0} \xi_\epsilon(F). \tag{1.2.1}$$

Let now $Sz(X) = \zeta(B_{X^*})$ and $\tau(X) = \xi(B_{X^*})$. The index Sz, which is usually called the Szlenk index, has been introduced by W. Szlenk in [Szl68]. The index τ is called the dentability index. It is clear that if $X \cong Y$, then $Sz(X) = Sz(Y)$ and $\tau(X) = \tau(Y)$. If Y is a subspace of X, then $Sz(Y) \leqslant Sz(X)$ and $Sz(Y) \leqslant Sz(X)$.
The following proposition is mentioned in [Bos02] and it is a consequence of Baire's theorem (see [DGZ93, Theorem I.5.2]).

Proposition 1.2.7. *Let X be a separable Banach space. The following assertion are equivalent:*
(i) X^* is separable,
(ii) $Sz(X) < \omega_1$,
(iii) $\tau(X) < \omega_1$.

Theorem 1.2.8. ***[Bos02, Theorem 4.11].*** *The indices Sz and τ are both co-analytic ranks on the family of all Banach with a separable dual space.*

1.3 Fréchet derivative

Definition 1.3.1. Let f be a real valued function defined on the Banach space X. We say that f is Gâteaux differentiable or Gâteaux smooth at $x \in X$, if for each $h \in X$,

$$f'(x)(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

exists and is a linear continuous function in h (i.e., $f'(x) \in X^*$). The functional $f'(x)$ is then called the Gâteaux derivative or Gâteaux differential of f at x. if, in addition, the above limit is uniform in $h \in S_X$, we say that f is Fréchet differentiable at x. Equivalently, f is Fréchet differentiable at x if there exists $f'(x) \in X^*$ such that

$$\lim_{y\to 0} \frac{f(x+y) - f(x) - f'(x)y}{\|y\|} = 0. \tag{1.3.1}$$

The functional $f'(x)$ is then called the Fréchet derivative of f at x.

We say that the norm $\|.\|$ of X is a Fréchet differentiable norm or a Fréchet smooth norm if $\|.\|$ is Fréchet differentiable at all $x \in S_X$.

Lemma 1.3.2. ***[DGZ93, Lemma I.1.3].*** *If $\|.\|$ denotes the norm of a Banach space X and $x \in S_X$, then the following are equivalent:*
(i) The norm $\|.\|$ is Fréchet differentiable at x.
(ii) $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ exists for every $h \in X$ and is uniform in $h \in S_X$.
(iii) $\lim_{\|y\|\to 0} \frac{\|x+y\|+\|x-y\|-2\|x\|}{\|y\|} = 0$.

Proof. (i) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (ii): A standard convexity argument implies that for fixed h, the quotient $\frac{\|x+th\|-\|x\|}{t}$ is a monotone function in t. Thus the one sided limits

$$\lim_{t\to 0^+} \frac{\|x+th\| - \|x\|}{t}$$

and

$$\lim_{t\to 0^-} \frac{\|x+th\| - \|x\|}{t}$$

always exist. If (iii) holds, these two limits are equal, since

$$\frac{\|x+th\| + \|x-th\| - 2\|x\|}{t} = \left(\frac{\|x+th\| - \|x\|}{t}\right) - \left(\frac{\|x-th\| - \|x\|}{-t}\right).$$

(ii) $\Rightarrow$ (i): Use subadditivity of the norm function and note that

$$\lim_{t\to 0^+} \frac{\|x+th\| - \|x\|}{t} \text{ is subadditive and} \leqslant \|h\|,$$

while

$$\lim_{t\to 0^-} \frac{\|x+th\| - \|x\|}{t} \text{ is supperadditive in } h$$

□

We have the following basic results, for the complete proof the reader is referred to [DGZ93].

Theorem 1.3.3. ***[DGZ93, Theorem I.1.4 (ŠMULYAN)]****. Suppose that $\|.\|$ is a norm on a Banach space X with dual norm $\|.\|_{X^*}$. Then*

(i) The norm $\|.\|$ is Fréchet differentiable at $x \in S_X$ if and only if whenever $f_n, g_n \in S_{X^*}$, $f_n(x) \to 1$ and $g_n(x) \to 1$, then $\|f_n - g_n\|_{X^*} \to 0$.

(ii) The norm $\|.\|_{X^*}$ is Fréchet differentiable at $f \in S_{X^*}$ if and only if whenever $x_n, y_n \in S_X$, $f(x_n) \to 1$ and $f(y_n) \to 1$, then $\|x_n - y_n\|_X \to 0$.

(iii) The norm $\|.\|$ is Gâteaux differentiable at $x \in S_X$ if and only if whenever $f_n, g_n \in S_{X^*}$, $f_n(x) \to 1$ and $g_n(x) \to 1$, then $f_n - g_n \xrightarrow{w^*} 0$.

(iv) The norm $\|.\|_{X^*}$ is Gâteaux differentiable at $f \in S_{X^*}$ if and only if whenever $x_n, y_n \in S_X$, $f(x_n) \to 1$ and $f(y_n) \to 1$, then $x_n - y_n \xrightarrow{w^*} 0$.

Corollary 1.3.4. ***[DGZ93, Corollary I.1.5]****. Let X be a Banach space and $\|.\|$ be a norm on X.*

(i) The norm $\|.\|$ is Gâteaux differentiable at $x \in S_X$ if and only if there is a unique $f \in S_{X^*}$ such that $f(x) = 1$ (we say that f is exposed in B_{X^*} by x or that x exposes f in B_{X^*}).

(ii) The norm $\|.\|$ is Fréchet differentiable at $x \in S_X$ if and only if there is a unique $f \in S_{X^*}$ satisfying:
for every $\epsilon > 0$ there exists $\delta > 0$ such that $g \in B_{X^*}$ and $g(x) > 1 - \delta$ imply $\|g - f\| < \epsilon$.
(we say that f is strongly exposed in B_{X^*} by x or that x strongly exposes f in B_{X^*})

Consider a functional $f \in X^*$. We say that it attains its supremum over C is there is $x \in C$ such that $f(c) = \sup\{f(x);\ c \in C\}$.

We say that $f \in X^*$ attains its norm if there is $b \in B_X$such that $f(b) = \|f\|$.

Theorem 1.3.5. ***[FHH⁺11, Theorem 7.14 (Bishop-Phelps Theorem)]****. Let C be a nonempty closed convex and bounded subset of a real Banach space X. Then the set of all continuous linear functionals on X that attain their maximum on C is dense in X^*. In Particular, the set of all continuous linear functionals on X that attain their norm (i.e., their maximum on B_X) is dense in X^*.*

1.4 The descriptive complexity of the family of Banach spaces with the bounded approximation property

In this section we will deal with the bounded approximation property and determine the descriptive complexity of the subset of $\mathcal{SE}$ containing the separable Banach spaces with the bounded approximation property. First, we need the definition of the bounded approximation property. We refer the reader to [Cas01]) or [LT77] for more about the approximation property and bounded approximation property.

Definition 1.4.1. [Cas01, Definition 2.2]. A Banach space X is said to have *the approximation property* (AP for short) if for every compact set K in X and every $\epsilon > 0$, there is a finite rank operator T so that $\|Tx - x\| \leqslant \epsilon$, for every $x \in K$.

Definition 1.4.2. [Cas01, Definition 3.1]. Let X be Banach space and $1 \leq \lambda < \infty$. We say that X has *the λ-bounded approximation property* (λ-BAP), if for every compact set K in X and every $\epsilon > 0$, there exists a finite rank operator $T : X \longrightarrow X$ with $\|T\| \leqslant \lambda$ and $\|T(x) - x\| < \epsilon$ for every $x \in K$. We say that X has the bounded approximation property (BAP for short) if X has the λ-BAP, for some λ. Finally, a Banach space is said to have *the metric approximation property* (MAP for short) if it has the 1-BAP.

The compact sets in Definition (1.4.2) can be replaced by finite sets. To see this, let K be a compact set in a Banach space X and let $\epsilon > 0$ be given. We can find a finite set $(x_i)_{i=1}^n$ so that $K \subset \bigcup_{i=1}^n B(x_i, \frac{\epsilon}{3\lambda})$. Now, if T is a finite rank operator with $\|Tx_i - x_i\| \leqslant \frac{\epsilon}{3}$, for $1 \leqslant i \leqslant n$, then $\|Tx - x\| \leqslant \epsilon$, for every $x \in K$.
One of the natural results on the BAP is that a separable Banach space has the BAP if and only if there is a uniformly bounded sequence (T_n) of finite rank operators on X which tends strongly to the identity on X. The sequence (T_n) is called an approximating sequence.

Grothendieck proved the surprising result that in many cases the approximation property implies the metric approximation property, for the proof see Theorem 1.e.15 in [LT77].

Theorem 1.4.3. ***[LT77, Theorem 1.e.15].*** *Let X be a separable space which is isometric to a dual space and which has the AP. Then X has the MAP.*

It follows from the preceding theorem that, for separable reflexive Banach spaces, the AP implies the MAP.

In order to determine the descriptive complexity of the set $\{X \in \mathcal{SE};\ X \textit{ has the BAP}\}$, we need the following structure which is equivalent to the λ-BAP for some $\lambda \geqslant 1$.

Lemma 1.4.4. *Suppose $(x_n)_{n=1}^\infty$ is a dense sequence in a Banach space X and $\lambda \geqslant 1$. Then X has the λ-bounded approximation property if and only if*

$$\forall K\ \forall \epsilon > 0\ \exists \lambda' < \lambda\ \exists R\ \forall N\ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R\ \forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$$

$$\|\sum_{i=1}^N \alpha_i [\sum_{j=1}^R \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^N \alpha_i x_i\|. \tag{1.4.1}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^R \sigma_i(j) x_j]\| \leqslant \epsilon \tag{1.4.2}$$

where K, R, N vary over $\mathbb{N}$ and ϵ, λ' vary over $\mathbb{Q}$.

Proof. Indeed, suppose X has the λ-bounded approximation property and let K and $\epsilon > 0$ be given. Then there is a finite rank operator $T : X \mapsto X$ of norm $< \lambda$ so that $\|z - T(z)\| < \epsilon$ for all z in the compact set $\{x_1, \cdots, x_K\}$. By perturbing T, we may suppose that T maps into the finite-dimensional subspace $[x_1, \cdots, x_R]$ for some R. Pick some $\|T\| < \lambda' < \lambda$. Then, for every N, we may perturb T slightly so that $\|T\| < \lambda'$ and that $T(x_i)$ belongs to the $\mathbb{Q}$-linear span of the $(x_j)_{j \leqslant R}$ for all $i \leqslant N$. Define now $\sigma_i \in \mathbb{Q}^R$ by $T(x_i) = \sum_{j=1}^R \sigma_i(j) x_j$ and let $\lambda' = \|T\|$. Then the two inequalities above hold for all $\alpha_1, \cdots, \alpha_N$ and $i \leqslant K$.

Conversely, suppose that the above criterion holds and that $C \subseteq X$ is compact and $\epsilon' > 0$. Pick a rational $\frac{\epsilon'}{3\lambda} > \epsilon > 0$ and a K so that every point of C is within ϵ of some x_i, $i \leqslant K$. So let λ' and R be given as above. Then, for every N and all $i \leqslant N$, define $y_i^N = \sum_{j=1}^R \sigma_i(j) x_j \in [x_1, \cdots, x_R]$, where the σ_i are given depending on N. We have that

$$\|\sum_{i=1}^{N}\alpha_i y_i^N\| \leqslant \lambda^{'} \cdot \|\sum_{i=1}^{N}\alpha_i x_i\|$$

for all $\alpha_i \in \mathbb{Q}$, and

$$\|x_i - y_i^N\| < \epsilon$$

for all $i \leqslant K$. In particular, for every i, the sequence $(y_i^N)_{N=i}^{\infty}$ is contained in a bounded set in a finite-dimensional space. So by a diagonal procedure, we may find some subsequence (N_l) so that $y_i = \lim_{l\to\infty} y_i^{N_l}$ exists for all i. By consequence

$$\|\sum_{i=1}^{\infty}\alpha_i y_i\| \leqslant \lambda^{'} \cdot \|\sum_{i=1}^{\infty}\alpha_i x_i\|$$

for all $\alpha_i \in \mathbb{Q}$, and

$$\|x_i - y_i\| \leqslant \epsilon$$

for all $i \leqslant K$.

Now, since the x_i are dense in X, there is a uniquely defined bounded linear operator $T : X \mapsto [x_1, \cdots, x_R]$ satisfying $T(x_i) = y_i$. Moreover, $\|T\| \leqslant \lambda^{'} < \lambda$ and $\|x_i - T(x_i)\| \leqslant \epsilon$ for all $i \leqslant K$. It follows that $\|z - T(z)\| < \epsilon^{'}$ for all $z \in C$. □

Now, we are ready to prove the main result in this section.

Theorem 1.4.5. *The set of all separable Banach spaces that have the BAP is a Borel subset of $\mathcal{SE}$.*

Proof. Let $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda^{'}$ vary over $\mathbb{Q}$, then we consider the set $E_{K,\epsilon,\lambda^{'},R,N,\sigma,\alpha} \subseteq C(\Delta)^{\mathbb{N}}$ such that:

$$E_{K,\epsilon,\lambda^{'},R,N,\sigma,\alpha} = \{(x_n)_{n=1}^{\infty} \in C(\Delta)^{\mathbb{N}};\ (1.4.1)\ \textit{and}\ (1.4.2)\ \textit{hold}\}$$

This set is closed in $C(\Delta)^{\mathbb{N}}$. Therefore, for $\lambda \in \mathbb{Q}$

$$E_{\lambda} = \bigcap_{K}\bigcap_{\epsilon}\bigcup_{\lambda^{'}<\lambda}\bigcup_{R}\bigcap_{N}\bigcup_{\sigma\in(\mathbb{Q}^R)^N}\bigcap_{\alpha\in\mathbb{Q}^N} E_{K,\epsilon,\lambda^{'},R,N,\sigma,\alpha}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. Moreover, the set

$$E = \bigcup_{\lambda\in\mathbb{Q}} E_{\lambda}$$

is also Borel.
There is a Borel map $d : \mathcal{SE} \longrightarrow C(\Delta)^{\mathbb{N}}$ such that $\overline{d(X)} = X$, by Theorem (1.2.1). Moreover, the previous Lemma implies that

$$X\ \textit{has the BAP} \iff d(X) \in E$$

Therefore, $\{X \in \mathcal{SE};\ X\ \textit{has the BAP}\}$ is a Borel subset of $\mathcal{SE}$. □

Also, our argument shows:

Proposition 1.4.6. *The map $\psi : \{X \in \mathcal{SE};\ X\ \textit{has the BAP}\} \longrightarrow [1,\infty[$, defined by $\psi(X) = \inf\{\lambda;\ X\ \textit{has the}\ \lambda - BAP\}$, is Borel.*

Proof. Let $\lambda^{'} \in \mathbb{Q}$, and $G_{\lambda^{'}} = \bigcap_{K}\bigcap_{\epsilon}\bigcup_{R}\bigcap_{N}\bigcup_{\sigma\in(\mathbb{Q}^R)^N}\bigcap_{\alpha\in\mathbb{Q}^N} E_{K,\epsilon,\lambda^{'},R,N,\sigma,\alpha}$. Then

$$\psi(X) \leqslant \lambda' \Longleftrightarrow d(X) \in G_{\lambda'},$$

and so

$$\{X \in \mathcal{SE};\ \psi(X) \leqslant \lambda'\}$$

is a Borel set. Thus, for all $\lambda' \in \mathbb{Q}$, $\psi^{-1}([1,\ \lambda'])$ is Borel. Therefore, ψ is a Borel map. $\square$

A sequence $\{e_i\}_{i=1}^{\infty}$ in the separable Banach space X is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars $\{a_i\}_{i=1}^{\infty}$, called the coordinates of x, such that $x = \sum_{i=1}^{\infty} a_i e_i$. In addition, a sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space X is called a basic sequence if $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis of $\overline{span}\ \{e_i\}_{i=1}^{\infty}$.
Since the sequence $\{e_i\}_{i=1}^{\infty}$ is a basic sequence if and only if there is $K > 0$ such that for all $n < m$ and scalars $a_1, \cdots, a_m$ we have: $\|\sum_{i=1}^{n} a_i e_i\| \leqslant K\|\sum_{i=1}^{m} a_i e_i\|$, (see e.g., [FHH+11] or [LT77]). Then we can prove the following result.

Proposition 1.4.7. *The set of all Banach spaces with a Schauder basis is an analytic subset of $\mathcal{SE}$.*

Proof. Let $R \in \mathbb{Q}$, $\alpha = (a_i) \in \mathbb{Q}^{<\mathbb{N}}$ and $n,\ m \in \mathbb{N}$, with $n < m \leqslant |\alpha|$. The set

$$A_{R,\,\alpha,\,m,\,n} = \{(x_i) \in C(\Delta)^{\mathbb{N}};\ \|\textstyle\sum_{j=1}^{n} a_i\, x_i\| \leqslant R\ \|\sum_{j=1}^{m} a_i\, x_i\|\}.$$

is closed in $C(\Delta)$. Therefore,

$$A = \bigcup_{R} \bigcap_{\alpha} \bigcap_{m \leqslant |\alpha|} \bigcap_{n \leqslant m} A_{R,\,a,\,m,\,n}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. By Lemma (1.2.2-(ii)), the set:

$$\Omega = \{((x_i), X);\ X = \overline{span}(x_i)\ \textit{and}\ (x_i) \in A\}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}} \times \mathcal{SE}$. Let $\pi:\ C(\Delta)^{\mathbb{N}} \times \mathcal{SE} \longrightarrow \mathcal{SE}$ be the canonical projection. Then, $\pi((A \times \mathcal{SE}) \bigcap \Omega) = \{X \in \mathcal{SE};\ X \textit{ has a basis}\}$ is an analytic subset of $\mathcal{SE}$. $\square$

It is unknown if the set of all separable Banach spaces with a basis is Borel or not. If it is non Borel set, then this will give a non trivial result that the set of all separable Banach spaces with the BAP and without a basis is co-analytic non-Borel. It has been shown by S. J. Szarek in [Sza87] that this set is not empty.

1.5 The descriptive complexity of the family of Banach spaces with the π-property

We refer the reader to the references [Cas01] and [CK91] for more information about the π-property. It was observed that if X has a basis, then it has the bounded approximation property given by the natural basis projection operators. This property was isolated, today, it is called the π-property.

Definition 1.5.1. [Cas01, Definition 5.1]. A Banach space X is said to have the π_λ-*property* if there is a net of finite rank projections (S_α) on X converging strongly to the identity on X with $\limsup_\alpha \|S_\alpha\| \leqslant \lambda$. A space with the π_λ-property for some λ is said to have the π-property.

It is clear that every space with a basis has the π-property given by the basis projections. Clearly, the π-property implies the BAP.
It turns out that a bounded operator which is sufficiently close (in norm) to its square will induce a projection on a Banach space.
The exact measure of closeness required was found by P. G. Casazza and N. J. Kalton in [CK91].

Proposition 1.5.2. ***[CK91, Proposition 3.6].*** *Suppose X is a Banach space and that T is bounded operator on X. Suppose $\|T - T^2\| = c < \frac{1}{4}$. Then there is a projection P on X such that $\{x : Tx = x\} \subset P(X) \subset T(X)$ and*

$$\|P\| \leqslant \frac{1}{2}\left(1 + \frac{1 + 2\|T\|}{(1-4c)^{\frac{1}{2}}}\right) \tag{1.5.1}$$

Proof. Define

$$S = \sum_{m=0}^{\infty} \binom{2m}{m} (T - T^2)^m$$

and

$$P = \frac{1}{2}(I - (I - 2T)S).$$

Since $(1-4z)^{-\frac{1}{2}}$ has a power series expansion $\sum_{m=0}^{\infty} \binom{2m}{m} z^m$ valid for $|z| < \frac{1}{4}$, then

$$\begin{aligned}
\|S\| &\leqslant \sum_{m=0}^{\infty} \binom{2m}{m} (\|T - T^2\|)^m \\
&\leqslant \sum_{m=0}^{\infty} \binom{2m}{m} c^m \\
&= (1-4c)^{-\frac{1}{2}}
\end{aligned}$$

where $\|T - T^2\| = c < \frac{1}{4}$, and by a power series manipulation that $(I - 2T)^2 S^2 = I$ Let $P = \frac{1}{2}(I - (I - 2T)S)$, then

$$\begin{aligned}
P^2 &= \frac{1}{4}(I - (I - 2T)S)^2 \\
&= \frac{1}{4}(I - 2(I - 2T)S + (I - 2T)^2 S^2) \\
&= \frac{1}{4}(I - 2(I - 2T)S + I) \\
&= \frac{1}{2}(I - (I - 2T)S) \\
&= P
\end{aligned}$$

Hence P is a projection on X and

$$\begin{aligned}
\|P\| &\leqslant \frac{1}{2}(1 + (1 + 2\|T\|)\|S\|) \\
&\leqslant \frac{1}{2}(1 + \frac{(1 + 2\|T\|)}{(1-4C)^{\frac{1}{2}}})
\end{aligned}$$

Moreover,

$$\begin{aligned}
P &= \frac{1}{2}(I-(I-2T)\sum_{m=0}^{\infty}\binom{2m}{m}(T-T^2)^m) \\
&= \frac{1}{2}(I-(I-2T)(I+2T-2T^2+\sum_{m=2}^{\infty}\binom{2m}{m}(T-T^2)^m)) \\
&= \frac{1}{2}(I-I-2T+2T-4T^2-2T^2+4T^3+(I-2T)\sum_{m=2}^{\infty}\binom{2m}{m}(T-T^2)^m) \\
&= 3T^2-2T^3+\frac{1}{2}(I-2T)\sum_{m=2}^{\infty}\binom{2m}{m}(T-T^2)^m.
\end{aligned}$$

For $x \in X$ such that $T(x) = x$ then $P(x) = x$ and clear that $P(X) \subset T(X)$ □

In view of Proposition (1.5.2) and the definition of the π-property we obtain:

Theorem 1.5.3. ***[CK91, Theorem 3.7]****. Let X be a separable Banach space. Suppose X has an approximating sequence T_n for which $\limsup_{n\to\infty}\|T_n - T_n^2\| < \frac{1}{4}$. Then X has the π-property.*

We need the following result to prove that the family of Banach spaces with the π-property is Borel.

Lemma 1.5.4. *Suppose $(x_n)_{n=1}^{\infty}$ is a dense sequence in a Banach space X. Then X has the π-property if and only if*

$\exists\lambda > 1\ \forall c \in (0, \frac{1}{4})\bigcap\mathbb{Q}\ \forall K\ \forall\epsilon > 0\ \forall\lambda' > \lambda\ \exists R\ \forall N \geqslant R\ \exists\sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R$ $\forall\alpha_1, \cdots, \alpha_N \in \mathbb{Q}$

$$\|\sum_{i=1}^{N}\alpha_i[\sum_{j=1}^{R}\sigma_i(j)x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N}\alpha_i x_i\|. \tag{1.5.2}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^{R}\sigma_i(j)x_j]\| \leqslant \epsilon \tag{1.5.3}$$

$$\|\sum_{i=1}^{N}\alpha_i[\sum_{j=1}^{R}\sigma_i(j)x_j] - \sum_{i=1}^{N}\alpha_i[\sum_{t=1}^{R}[\sum_{j=1}^{R}\sigma_i(j)\sigma_j(t)]x_t]\| \leqslant c \cdot \|\sum_{i=1}^{N}\alpha_i x_i\|. \tag{1.5.4}$$

where K, R, N vary over $\mathbb{N}$ and ϵ, λ', and λ vary over $\mathbb{Q}$.

Proof. Indeed, suppose X has the π_λ -property, then there exists a sequence (P_n) of finite rank projections such that $\|P_n\| < \lambda$, for all n and P_n converges strongly to the identity. By perturbing P_n, we may suppose that P_n maps into the finite-dimensional subspace $[x_1, \cdots, x_{R_n}]$ for some R_n in $\mathbb{N}$ but we still have (1.5.3) and $\|P_n\| < \lambda$. Then, for every N, we may perturb P_n slightly so that $\|P_n\| < \lambda$ and $P_n(x_i)$ belongs to the $\mathbb{Q}$-linear span of the x_j for all $i \leqslant N$, such that (1.5.2), (1.5.3) and (1.5.4) still hold. Define now $(\sigma_i^{(n)}) \in (\mathbb{Q}^{R_n})^N$, such that $P_n(x_i) = \sum_{j=1}^{R_n}\sigma_i^{(n)}(j)x_j$. Since $P_n^2(x_i) = \sum_{t=1}^{R_n}[\sum_{j=1}^{R_n}\sigma_i(j)\sigma_j(t)]x_t$, the three inequalities hold for all $\alpha_1, \cdots, \alpha_N \in \mathbb{Q}$, and $i \leqslant K$.

Conversely, suppose that the above criterion holds and that $\epsilon' > 0$. Pick a rational $\frac{\epsilon'}{3\lambda} > \epsilon > 0$ and a K. So let λ' and R be given as above. Then for every N and $i \leqslant N$, define $y_i^N = \sum\limits_{j=1}^{R} \sigma_i(j) x_j$, and $z_i^N = \sum\limits_{t=1}^{R} [\sum\limits_{j=1}^{R} \sigma_i(j)\sigma_j(t)] x_t$ in $[x_1, \cdots x_R]$, where the σ_i are given depending on N. We have that

$$\|\sum_{i=1}^{N} \alpha_i y_i^N\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{1.5.5}$$

for all $\alpha_i \in \mathbb{Q}$,

$$\|x_i - y_i^N\| \leqslant \epsilon \tag{1.5.6}$$

for all $i \leqslant K$, and

$$\|\sum_{i=1}^{N} \alpha_i y_i^N - \sum_{i=1}^{N} \alpha_i z_i^N\| \leqslant c \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{1.5.7}$$

for all $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$. In particular, for every i, the sequences $(y_i^N)_{N=i}^{\infty}$, and $(z_i^N)_{N=i}^{\infty}$ are contained in a bounded set in a finite-dimensional space. So by a diagonal procedure, we may find some subsequence (N_l) so that $y_i = \lim\limits_{l \to \infty} y_i^{N_l}$ and $z_i = \lim\limits_{l \to \infty} z_i^{N_l}$ exists for all i. By consequence

$$\|\sum_{i=1}^{\infty} \alpha_i y_i\| \leqslant \lambda' \cdot \|\sum_{i=1}^{\infty} \alpha_i x_i\|. \tag{1.5.8}$$

for all $\alpha_i \in \mathbb{Q}$,

$$\|x_i - y_i\| \leqslant \epsilon \tag{1.5.9}$$

for all $i \leqslant K$, and

$$\|\sum_{i=1}^{\infty} \alpha_i y_i - \sum_{i=1}^{\infty} \alpha_i z_i\| \leqslant c \cdot \|\sum_{i=1}^{\infty} \alpha_i x_i\|. \tag{1.5.10}$$

for all $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$.

Now, since the x_i are dense in X, there are uniquely defined bounded linear operators $T_{K,\epsilon} : X \mapsto [x_1, \cdots, x_R]$ satisfying $T_{K,\epsilon}(x_i) = y_i$ and then $T_{K,\epsilon}^2 : X \mapsto [x_1, \cdots, x_R]$ satisfies $T_{K,\epsilon}^2(x_i) = z_i$ such that $\|T_{K,\epsilon}\| \leqslant \lambda' < \lambda$ and $\|x_i - T_{K,\epsilon}(x_i)\| \leqslant \epsilon$ for all $i \leqslant K$. Let $K \to \infty$ and $\epsilon \to 0$, then $T_{K,\epsilon}(x_i) \to x_i$ for all $x_i \in (x_i)$ strongly. Since (x_i) is a dense sequence in X and the operators $T_{K,\epsilon}$ are uniformly bounded, then $T_{K,\epsilon}(x) \to x$ for all $x \in X$ strongly. Also, $\lim\sup \|T_{K,\epsilon} - T_{K,\epsilon}^2\| = c < \frac{1}{4}$. Therefore, by Theorem 1.5.3, X has the $\pi_{\lambda+1}$-property as $c \to 0$. □

Finally, we prove the main result in this section depending on the previous results.

Theorem 1.5.5. *The set of all separable Banach spaces that have the π-property is a Borel subset of $\mathcal{SE}$.*

Proof. Let $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda'$ vary over $\mathbb{Q}$. Let also $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$, $\sigma \in (\mathbb{Q}^R)^N$, and $\alpha \in \mathbb{Q}^N$, then we consider the set $E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha} \subseteq C(\Delta)^{\mathbb{N}}$ such that:

$$E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha} = \{(x_n)_{n=1}^{\infty} \in C(\Delta)^{\mathbb{N}};\ (1.5.2)\ (1.5.3)\ \textit{and}\ (1.5.4)\ \textit{hold}\}$$

This set is closed in $C(\Delta)^{\mathbb{N}}$. Therefore, for $\lambda \in \mathbb{R}$

$$E_\lambda = \bigcap_{c} \bigcap_{K} \bigcap_{\epsilon} \bigcap_{\lambda' > \lambda} \bigcup_{R} \bigcap_{N} \bigcup_{\sigma \in (\mathbb{Q}^R)^N} \bigcap_{\alpha \in \mathbb{Q}^N} E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. Moreover, the set

$$E = \bigcup_{\lambda \in \mathbb{Q}} E_\lambda$$

is also Borel.
There is a Borel map $d : \mathcal{SE} \longrightarrow C(\Delta)^{\mathbb{N}}$ such that $\overline{d(X)} = X$, by Theorem (1.2.1). Moreover, the previous Lemma implies that

$$X \text{ has the } \pi - \text{property} \iff d(X) \in E.$$

Therefore, $\{X \in \mathcal{SE};\ X \text{ has the } \pi - \text{property}\}$ is a Borel subset of $\mathcal{SE}$. $\square$

1.6 The descriptive complexity of the family of reflexive Banach spaces with a FDD

Having an approximation sequence (T_n) whose elements commute yields to the approximation property. The converse is open question for the separable Banach spaces, for more see [Cas01].

Definition 1.6.1. [Cas01, Definition 4.1]. A Banach space X has the λ-*commuting bounded approximation property* (λ-CBAP) if there is a net (T_α) of finite rank operators on X converging strongly to the identity such that $\limsup_\alpha \|T_\alpha\| \leqslant \lambda$ and for all α, β wee have $T_\alpha T_\beta = T_\beta T_\alpha$. We say that X has the commuting bounded approximation property (CBAP) if it has the λ-commuting bounded approximation property for some $\lambda \geqslant 1$. A Banach space is said to have the commuting metric approximation property (CMAP) if it has 1-CBAP.

It is clear from the definition that the CBAP implies the BAP. The converse is an open question, but it is not the same for the MAP. Casazza and Kalton proved the following theorem in [CK91] that the MAP implies the CMAP for separable Banach spaces, but first we need the next corollary:

Corollary 1.6.2. ***[CK91, Corollary 2.2.].*** *Suppose X has an approximating sequence T_n for which*

$$\lim_{m,n\to\infty} \|T_m T_n - T_n T_m\| = 0$$

and $\liminf_{n\to\infty} \|T_n\| = \lambda$. *Then X has λ-CBAP.*

Theorem 1.6.3. ***([CK91, Theorem 2.4]).*** *Suppose X is a separable Banach space with a MAP. Then X has the CMAP.*

Proof. We state the proof in order to completeness. Suppose that the Banach space X has an approximating sequence T_n with $T_m T_n = T_n$ for $m > n$ and $\|T_n\| \leqslant 1 + \epsilon_n$ where $\sum \epsilon_n = \beta < \infty$. For $t > 0$ define the linear operators

$$V_n(t) = e^{-nt} \exp\,(t \sum_{k=1}^{n} T_k) = e^{-nt} \sum_{j=0}^{\infty} \frac{t^j}{j!} (T_1 + \cdots + T_n)^j.$$

Then,

$$\begin{aligned}
\|V_n(t)\| &\leqslant e^{-nt}\sum_{j=0}^{\infty}\frac{t^j}{j!}(\|T_1\|+\cdots+\|T_n\|)^j \\
&\leqslant e^{-nt}\sum_{j=0}^{\infty}\frac{t^j}{j!}(\sum_{k=1}^{n}(1+\epsilon_k))^j \\
&= e^{-nt}\sum_{j=0}^{\infty}\frac{t^j}{j!}(n+\sum_{k=1}^{n}\epsilon_k)^j \\
&\leqslant e^{-nt}\sum_{j=0}^{\infty}\frac{t^j}{j!}(n+\beta)^j \\
&= e^{-nt}e^{tn+t\beta} \\
&= e^{t\beta}
\end{aligned}$$

Let $E_n = T_n(X)$. Then each E_n is an invariant subspace for every T_m and hence also for every $V_m(t)$. Rewriting $V_m(t)$ as $\exp(t\sum_{k=1}^{m}(T_k - I))$. If $x \in E_n$ and $m > n$, then

$$\begin{aligned}
V_m(x) &= e^{-nt}\exp t\sum_{k=1}^{m}(T_k(x)-(x)) \\
&= e^{-nt}\exp t\sum_{k=1}^{n}(T_k(x)-(x)),\ \textit{because } T_k(x)=x,\ \forall x\in E_n,\ \textit{and } \forall k>n \\
&= V_n(t)(x)
\end{aligned}$$

It follows therefore from the bound on the norms of $V_n(t)$ that we can define $S(t)$ by $S(t)x = \lim_{n\to\infty} V_n(t)x$ for all $x \in X$. Clearly $\|S(t)\| \leqslant e^{\beta t}$. furthermore, $S(t)$ has the semigroup property $S(t_1+t_2) = S(t_1)S(t_2)$ since each $V_n(t)$ is a semigroup and the property is preserved by strong limits.
Furthermore, $S(t)$ is compact for $t > 0$. Indeed, suppose $l \in \mathbb{N}$ and that $x \in E_n$ where $n > l$. Then $d(S(t)x, E_l) = d(V_n(t)x, E_l)$. It is then easy to see, by expansion, that the operator $\exp t(T_1+\cdots+T_n) - \exp t(T_{l+1}+\cdots+T_n)$ has range contained in E_l. Thus

$$\begin{aligned}
d(S(t)x, E_l) &= e^{-nt}d(\exp t(T_{l+1}+\cdots+T_n)x, E_l) \\
&\leqslant e^{-nt}\|\exp t(T_{l+1}+\cdots+T_n)\|\|x\| \\
&\leqslant e^{-nt}\exp t(\|T_{l+1}\|+\cdots+\|T_n\|)\|x\| \\
&\leqslant e^{\beta t}e^{-lt}\|x\|.
\end{aligned}$$

Hence for all $x \in X$,

$$d(S(t)x, E_l) \leqslant e^{\beta t}e^{-lt}\|x\|$$

and hence $S(t)$ is compact.
Note that as $t \to 0$, then $\|S(t)\| \to 1$. Also if $x \in E_n$ we have $S(t)x = V_n(t)x \to x$. Hence for all $x \in X$, we have $\lim_{t\to 0} S(t)x = x$.
Since X has MAP then there exist finite rank operators R_n so that $\|R_n - S(1/n)\| \to 0$. Then R_n is an approximating sequence, $\lim \|R_n\| = 1$ and $\lim_{m,n\to\infty}\|R_mR_n - R_nR_m\| = 0$ since the operators $S(1/n)$ commute. Hence by Corollary (1.6.2) the space X has the CMAP. □

Also, we need [Cas01, Theorem 4.9] for Banach spaces with separable dual:

Theorem 1.6.4. . *Let X be a Banach space with a separable dual. The following statements are equivalent:*

1. *X^* has the bounded approximation property.*
2. *X embeds complementably into a space with a shrinking basis.*
3. *X has the shrinking CBAP. That is, there is a commuting approximating sequence (T_n) on X with (T_n^*) converging strongly to the identity on X^*.*

A Schauder basis decomposes a Banach space into a direct sum of one-finite dimensional subspaces. A cruder form of this is to decompose the space into finite dimensional subspaces.

Definition 1.6.5. [Cas01, Definition 6.1]. A sequence (E_n) of finite dimensional subspaces of a Banach space X is called a (unconditional) finite decomposition for X (FDD(UFDD)) if for every $x \in X$ there is unique sequence $x_n \in E_n$ so that $x = \sum x_n$ (and this series converges unconditionally). In this case we will write $X = \sum_n \bigoplus E_n$ and say X has a FDD.

The separable Banach space X has a FDD if and only if there is an approximating sequence (P_n) of commuting projections on X, and in this case the finite dimensional decomposition is $X = \sum_n \bigoplus (P_n - P_{n-1})(X)$ where $P_0 = 0$.

Theorem 1.6.6. ***[Cas01, Theorem 6.3].*** *A separable Banach space has a finite dimensional decomposition (a FDD) if and only if it has both the commuting bounded approximation property (CBAP) and the π-property.*

We will now prove, with some work, that in some natural classes the existence of a finite-dimensional decomposition happens to be a Borel condition. We first consider the class of reflexive spaces.
The commuting bounded approximation property (CBAP) implies the bounded approximation property (BAP) by the definition of the CBAP. By Grothendieck's theorem (Theorem (1.4.3)) the BAP and the metric approximation property (MAP) are equivalent for reflexive Banach spaces. In addition, Theorem (1.6.3) implies that for any reflexive Banach space the CBAP is equivalent to MAP. For the set $\mathcal{R}$ of all separable reflexive Banach spaces, Theorem (1.5.5), Theorem (1.4.5) and Theorem (1.6.6) imply that there exists a Borel subset $B = \{X \in \mathcal{SE};\ X \ has\ the\ MAP\ and\ the\ \pi - property\}$ such that $\{X \in \mathcal{R}; X\ has\ a\ FDD\} = B \cap \mathcal{R}$.

We will extend this simple observation to some classes of non reflexive spaces. The following result has been proved in [Joh72]. The proof below follows the lines of [GS88].

Proposition 1.6.7. *Let X be a Banach space with separable dual. If X has the MAP for all equivalent norms then X^* has the MAP.*

Proof. Since X^* is separable, there is an equivalent Fréchet differentiable norm on X. If $\|.\|_X$ is a Fréchet differentiable norm and $x \in S_X$, there exists a unique $x^* \in S_{X^*}$ such that $x^*(x) = 1$, and x^* is a strongly exposed point of B_{X^*}. Since by assumption X equipped with this norm has the MAP, there exists an approximating sequence (T_n) with $\|T_n\| \leqslant 1$, and then for all $x^* \in X^*$ we have $T_n^*(x^*) \overset{w^*}{\to} x^*$. For all $x^* \in X^*$ which attains its norm we have $\|T_n^*(x^*) - x^*\|_{X^*} \to 0$. Bishop-Phelps theorem yields that for all $x^* \in X^*$, $\|T_n^*(x^*) - x^*\|_{X^*} \to 0$. Therefore, X^* has the MAP. $\square$

The set $\mathcal{SD}$ of all separable Banach spaces with separable dual spaces is co-analytic in $\mathcal{SE}$ and the Szlenk index Sz is a co-analytic rank on $\mathcal{SD}$. In particular, the set $S_\alpha = \{X \in \mathcal{SE};\ Sz(X) \leqslant \alpha\}$ is Borel in $\mathcal{SE}$. In this Borel set, the following holds.

Theorem 1.6.8. *The set of all separable Banach spaces in S_α that have a shrinking FDD is Borel in $\mathcal{SE}$.*

Proof. Indeed, by Theorem 1.2.5, we have that

$$S^*_\alpha = \{Y \in \mathcal{SE};\ \exists X \in S_\alpha \text{ } with \text{ } Y \simeq X^*\}$$

is analytic. Since the set $\{Y \in \mathcal{SE};\ Y \text{ } has \text{ } the \text{ } BAP\}$ is Borel by Theorem 1.4.5. Then

$$G^*_\alpha = \{Y \in \mathcal{SE};\ \exists X \in S_\alpha \text{ } with \text{ } Y \simeq X^* \text{ } and \text{ } Y \text{ } has \text{ } the \text{ } BAP\}$$

is analytic. By Proposition 1.2.6, we have that

$$G_\alpha = \{X \in S_\alpha;\ \exists Y \in G^*_\alpha, \text{ } with \text{ } Y \simeq X^*\}$$

is analytic.
Since $\{(X, Z); Z \simeq X\}$ is analytic in $\mathcal{SE} \times \mathcal{SE}$, and $\{Z;\ Z \text{ } fails \text{ } the \text{ } MAP\}$ is Borel. The set $\{(X, Z); Z \simeq X,\ Z \text{ } fails \text{ } the \text{ } MAP\}$ is analytic. Thus its canonical projection $\{X \in \mathcal{SE};\ \exists Z \in \mathcal{SE}; Z \simeq X,\ Z \text{ } fails \text{ } the \text{ } MAP\}$ is analytic. Now, Proposition 1.6.7 implies that the set

$$\begin{aligned} H_\alpha =& \{X \in S_\alpha;\ X^* \text{ } fails \text{ } the \text{ } AP\} \\ =& \{X \in S_\alpha;\ \exists Z \text{ } with \text{ } Z \simeq X \text{ } and \text{ } Z \text{ } fails \text{ } the \text{ } MAP\} \end{aligned}$$

is analytic. Since $S_\alpha \setminus H_\alpha = G_\alpha$ and both G_α and H_α are analytic sets in $\mathcal{SE}$, then both are Borel by the separation theorem. Now, Theorem 1.6.4 implies that $G_\alpha = \{X \in S_\alpha;\ X \text{ } has \text{ } the \text{ } shrinking \text{ } CBAP\}$. Thus,

$$\{X \in S_\alpha;\ X \text{ } has \text{ } a \text{ } shrinking \text{ } FDD\}$$

is Borel by Theorem 1.5.5 and Theorem 1.6.6. □

Questions: As seen before, a separable Banach space has CBAP if and only if it has an equivalent norm for which it has MAP. It follows that the set $\{X \in \mathcal{SE}; X \text{ } has \text{ } the \text{ } CBAP\}$ is analytic. It is not clear if it is Borel or not. Also, it is not known if there is a Borel subset B of $\mathcal{SE}$ such that $\{X \in \mathcal{SD};\ X^* \text{ } has \text{ } the \text{ } AP\} = B \cap \mathcal{SD}$. This would be an improvement of Theorem (1.6.8). Finally, what happens when we replace FDD by basis is not clear: for instance, the set of all spaces in S_α which have a basis is clearly analytic. Is it Borel?
We need the following result of [FG12].

Lemma 1.6.9. ***[FG12, Lemma 2.3]***. *Let A be a subset of 2^ω. The following assertions are equivalent:*

(1) A is comeager,
(2) there is a sequence $I_0 < I_1 < I_2 < \cdots$ of successive subsets of ω, and $a_n \subset I_n$, such that for any $u \in 2^\omega$, if the set $\{n;\ u \bigcap I_n = a_n\}$ is infinite, then $u \in A$.

Some works have been done on the relation between Baire Category and families of subspaces of a Banach space with a Schauder basis, (see e.g. [FG12]). In order to state the next theorem we recall that a fundamental and total biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called a Markushevich basis (M-basis) for $(X, \|.\|)$. Furthermore, a biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called λ- norming if $|||x||| = sup\{|x^*(x)|;\ x^* \in B_{X^*} \bigcap \overline{span}^{\|.\|}\{e_i^*\}_{i=1}^{\infty}\}$, $x \in X$, is a norm satisfying $\lambda||x|| \leqslant |||x|||$ for some $0 < \lambda \leqslant 1$, (see [HZSV07]). Also, any separable Banach space has a bounded norming M-basis, (see [Ter94]). We recall that a separable Banach space X has a finite dimensional decomposition (a FDD) $\{X_n\}_{n=1}^{\infty}$, where X_n's are finite dimensional subspaces of X, if for every $x \in X$ there exists a unique sequence (x_n) with $x_n \in X_n$, $n \in \mathbb{N}$, such that $x = \sum\limits_{n=1}^{\infty} x_n$.

Theorem 1.6.10. *Let X be a separable Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^{\infty}$. If we let $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u$ has a FDD$\}$ is comeager in the Cantor space Δ.*

Proof. We may and do assume that the M-basis is 1-norming. We will consider $\{I_j\}_{j=0}^{\infty}$ successive intervals of $\mathbb{N}$ where, $I_0 = \emptyset$ and for $j \geqslant 1$ we will write $I_j = F_j \cup G_j$, such that $F_j \cap G_j = \emptyset$, and $F_j = [min\ I_j,\ min\ I_j + \rho_j]$, such that $\rho_j \in \mathbb{N}$ and $\rho_j \leqslant (max\ I_j - min\ I_j)$. We can construct the sequence $\{I_j\}_{j=0}^{\infty}$ inductively satisfying the following property:

for $n \in \mathbb{N}$ and $x \in span\{e_i;\ i \in \bigcup\limits_{j=0}^{n-1} I_j \bigcup F_n\}$

$$\| x \| \leq (1 + \frac{1}{n}) \sup\{|x^*(x)|;\ \|x^*\| = 1,\ and\ x^* \in span\{e_i^*;\ i \in \bigcup_{j=0}^{n} I_j\}\} \tag{1.6.1}$$

Indeed, let $H_n = span\{e_i;\ i \in \bigcup\limits_{j=0}^{n-1} I_j \bigcup F_n\}$, then for any $\epsilon > 0$, S_{H_n} has an $\frac{\varepsilon}{2} - net$ finite set $A_n = \{y_{t,n}\}_{t=1}^{m_n}$ [such that (m_n) is an increasing sequence in $\mathbb{N}$ and $m_n = |A_n|$]. Moreover, we can choose for every $y_{t,n} \in A_n$, a functional map $y_{t,n}^* \in S_{X^*} \bigcap span\{e_i^*;\ i \geq 1\}$ where

$$|y_{t,n}^*(y_{t,n})| > 1 - \tfrac{\epsilon}{2}$$

Therefore, for any $x \in S_{H_n}$, there exists $y_{t,n} \in A_n$ such that:

$$||y_{t,n} - x|| < \tfrac{\epsilon}{2}$$

Then,

$$|y_{t,n}^*(y_{t,n}) - y_{t,n}^*(x)| < \frac{\epsilon}{2} \tag{1.6.2}$$

$$|y_{t,n}^*(x)| > 1 - \epsilon \tag{1.6.3}$$

Therefore,

$$||x|| < (1 - \epsilon)^{-1} |y_{t,n}^*(x)|,\ \forall x \in H_n \tag{1.6.4}$$

In particular, for $\epsilon = \frac{1}{n+1}$ we have the functional maps $\{y_{t,n}^*\}_{t=1}^{m_n}$ such that (2.6) holds. In order to complete our construction, let $supp(y_{t,n}^*) = \{i;\ y_{t,n}^*(e_i) \neq 0\}$, then $\bigcup\limits_{t=1}^{m_n} supp(y_{t,n}^*) = B_n$ is a finite set. Now, we can choose G_n to be a finite set such that

$$B_n \subseteq (\bigcup_{j=0}^{n-1} I_j \bigcup F_n \bigcup G_n) = \bigcup_{j=0}^{n} I_j$$

Suppose $u \in \Delta$ and $\{F_{n_k}\}$ is an infinite subsequence such that $u \bigcap I_{n_k} = F_{n_k}$ for all $k \in \mathbb{N}$. If $x \in \overline{span}\{e_i;\ i \in u\}$, then $x \in \{e_i^*;\ i \in G_{n_k}\}^{\perp}$, for all $k \in \mathbb{N}$. And so, $x = x_{n_k} + y$, where $x_{n_k} \in span\{e_i;\ i \in \Gamma_{n_k}\}$ such that $\Gamma_{n_k} = u \bigcap (\bigcup_{j=0}^{n_k-1} I_j \bigcup F_{n_k})$, and $y \in span\{e_i;\ i \in u \backslash \Gamma_{n_k}\}$. Define for every k a linear projection $P_{n_k} :\ span\{e_i;\ i \in u\} \longrightarrow\ span\{e_i;\ i \in u\}$, such that $P_{n_k}(x) = \sum_{i \in \Gamma_{n_k}} e_i^*(x) e_i$. Then

$$\begin{aligned}
\| P_{n_k}(x) \| &\leq (1 + \frac{1}{n_k}) \sup\{|x^*(P_{n_k}(x))|;\ \|x^*\| = 1,\ and\ x^* \in span\{e_i^*;\ i \in \bigcup_{j=0}^{n_k} I_j\}\} \\
&\leq (1 + \frac{1}{n_k}) \sup\{|x^*(x)|;\ \|x^*\| = 1,\ and\ x^* \in span\{e_i^*;\ i \in \bigcup_{j=0}^{n_k} I_j\}\} \\
&\leq (1 + \frac{1}{n_k}) \| x \|
\end{aligned}$$

Therefore, $\{P_{n_k}\}_{k=1}^{\infty}$ is a sequence of bounded linear projections such that $\sup_k \| P_{n_k} \| < \infty$, and $P_{n_{k_2}} P_{n_{k_1}} = P_{\min(n_{k_1}, n_{k_2})}$. Hence, these projections determine a finite dimensional decomposition (a FDD) of the subspace $E_u = \overline{span}\{e_i;\ i \in u\}$ by putting $E_u^1 = P_{n_1}(E_u)$ and $E_u^k = (P_{n_k} - P_{n_{k-1}})(E_u)$, for $k > 1$.
Therefor, E_u has a FDD if $\{n;\ u \bigcap I_n = F_n\}$ is infinite, thus $\{u \in \Delta;\ E_n\ has\ a\ FDD\}$ is comeager, by Lemma (1.6.9). $\square$

Note that since every space with a FDD has the BAP. Theorem (1.6.10) shows that $\{u \in \Delta;\ E_n\ has\ the\ BAP\}$ is comeager in Δ. We recall that an example of a separable Banach space with the BAP but without a FDD has been constructed by C. Read, (see [CK91]).

Chapter 2

Non-isomorphic Complemented Subspaces of reflexive Orlicz Function Spaces $L^{\Phi}[0,1]$

Introduction

Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. Consider the Cantor group $G = \{-1,1\}^{\mathcal{C}}$ equipped with the Haar measure. The dual group is the discrete group formed by Walsh functions $w_F = \prod_{c \in F} r_c$ where F is a finite subset of $\mathcal{C}$ and r_c is the Rademacher function, that is $r_c(x) = x(c)$, $x \in G$. These Walsh functions generate $L^p(G)$ for $1 \leq p < \infty$, and the reflexive Orlicz function spaces $L^{\Phi}(G)$, where Φ is an Orlicz function.

A measurable function f on G only depends on the coordinates $F \subset \mathcal{C}$, provided $f(x) = f(y)$ whenever $x, y \in G$ with $x(c) = y(c)$ for all $c \in F$. A measurable subset S of G depends only on the coordinates $F \subset \mathcal{C}$ provided χ_S does. Moreover, For $F \subset \mathcal{C}$ the sub-σ-algebra $\mathfrak{G}(F)$ contains all measurable subsets of G that depend only on the F-coordinates. A branch in $\mathcal{C}$ will be a subset of $\mathcal{C}$ consisting of mutually comparable elements. For more the reader is referred to [BRS81], [Bou81] and [DK14].

In [BRS81], the authors considered the subspace $X^p_{\mathcal{C}}$ which is the closed linear span in $L^p(G)$ over all finite branches Γ in $\mathcal{C}$ of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. In addition, they proved that $X^p_{\mathcal{C}}$ is complemented in $L^p(G)$ and isomorphic to L^p, for $1 < p < \infty$. Moreover, for a tree T on $\mathbb{N}$, the space X^p_T is the closed linear span in $L^p(G)$ over all finite branches Γ in T of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. Hence, X^p_T is a one-complemented subspace of $X^p_{\mathcal{C}}$ by the conditional expectation operator with respect to the sub-σ-algebra $\mathfrak{G}(T)$ which contains all measurable subsets of G that depend only on the T-coordinates. J. Bourgain in [Bou81] showed that the tree T is well founded if and only if the space X^p_T does not contain a copy of L^p, for $1 < p < \infty$ and $p \neq 2$. Consequently, it was shown that if B is a universal separable Banach space for the elements of the class $\{X^p_T;\ T \textit{ is a well founded tree}\}$, then B contains a copy of L^p. It follows that there are uncountably many mutually non-isomorphic members in this class.

In this note we will show that these results extend to the case of the reflexive Orlicz function spaces $L^{\Phi}[0,1]$, where Φ is an Orlicz function. Moreover, some of the results extend to rearrangement invariant function spaces under some conditions on the Boyd indices.

2.1 Rearrangement invariant function spaces

Many of the lattices of measurable functions which appear in analysis have an important symmetry property, namely they remain invariant if we apply a measure preserving transformation to the underlying measure space. Such lattices are called rearrangement invariant function spaces or r.i. function spaces, in short.

We shall restrict our attention to the case in which (Ω, Σ, μ) is a separable measure space (i.e., Σ with the metric $d(\sigma_1, \sigma_2) = \mu(\sigma_1 \Delta \sigma_2)$, is a separable metric space, where $\sigma_1 \Delta \sigma_2 = (\sigma_1 \setminus \sigma_2) \cup (\sigma_2 \setminus \sigma_1)$, for every $\sigma_1, \sigma_2 \in \Sigma$). The structure of such a measure space consists of continuous part which is isomorphic to the usual Lebesgue measure space on finite or infinite interval on the line and of an at most countable numbers of atoms. By an isomorphism of two measure spaces we mean one-to-one correspondence between the σ-algebras which preserves the measure and the countable Boolean operations. In general this correspondence is induced by a point transformation between the measure spaces. Since an automorphism τ of a measure space Ω (i.e., an invertible transformation τ from Ω onto itself so that, for any measurable subset σ of Ω, $\mu(\tau^{-1}\sigma) = \mu(\sigma)$) maps the continuous part of Ω into itself and maps each atom of Ω to an atom with the same mass it is clear that the study r.i. spaces over a separable measure space reduces immediately to the study of such spaces when Ω is either a finite or infinite interval on the line or a finite or countably infinite discrete measure space in which each point has the same mass. Thus we are reduced to the study of the following three cases:

1. $\Omega =$ integers and the mass of every point is one.
2. $\Omega = [0,1]$ with the usual Lebesgue measure.
3. $\Omega = [0,\infty)$ with the usual Lebesgue measure.

For the r.i function space X, the norm of an $f \in X$ will be assumed to depend only on the *distribution function*

$$d_f(t) = \mu(\{\omega \in \Omega; f(\omega) > t\}), \quad -\infty < t < \infty$$

of f or, in fact, on the distribution function of $|f|$. More precisely, if $f \in X$ and g is a measurable function such that $d_{|g|}(t) = d_{|f|}(t)$, for every $t \geqslant 0$ (i.e. $|f|$ and $|g|$ are μ-equimeasurable) then also $g \in X$ and $\|g\| = \|f\|$. The distribution function of non-negative function f is clearly a right continuous non-decreasing function on $[0,\infty)$. Of special importance in the investigation of r.i. spaces is the right continuous inverse f^* of d_f (for $f \geqslant 0$) which is defined by

$$f^*(s) = \inf\{t > 0; d_f(t) \leqslant s\}, \quad 0 \leqslant s < \mu(\Omega)$$

The function f^*, which is evidently non-increasing, right continuous and has the same distribution function as f, is called the *decreasing rearrangement* of f.
Since we deal mostly with separable r.i. function spaces there is no loss of generality in considering only the canonical cases of r.i. function spaces on $[0,1]$ and $[0,\infty)$.
A rearrangement invariant function space $X[0,1]$ on the interval $I = [0,1]$ is a Banach space of equivalent classes of measurable functions on I such that:

(i) $X[0,1]$ is a Banach lattice with respect to the pointwise order.

(ii) For every automorphism τ of I and every $f \in X[0,1]$, also $f(\tau) \in X[0,1]$ and $\|f(\tau)\| = \|f\|$.

(iii) $L^\infty(I) \subset X[0,1] \subset L^1(I)$, with norm one embeddings.

(iv) $L^\infty(I)$ is dense in $X[0,1]$.

In the presence of condition (i), (ii), (iv), condition (iii) simply means that the function $f \equiv 1$ has norm one in $X[0,1]$. Condition (iii) is meant to exclude trivial cases, e.g., when $X[0,1]$ consists only of constant functions. Often r.i. function spaces are defined without imposing condition (iv), in which case one has to distinguish between minimal r.i. function spaces; i.e., those satisfying also (iv), and maximal r.i. function spaces; i.e., those which do not have any r.i. enlargement. Since the separability of the r.i. function space X implies that (iv) is necessarily satisfied, we have included this condition as an axiom.

A r.i. function space $X[0,\infty)$ on the interval $I = [0,\infty)$ is a Banach space of measurable functions on I which satisfies (i) and (ii) and, instead of (iii) and (iv), the following two conditions

(iii′) $L^1[0,\infty) \bigcap L^\infty[0,\infty) \subset X[0,\infty) \subset L^1[0,\infty) + L^\infty[0,\infty)$ with norm one embeddings.

(iv′) For $[0,\infty)$, the simple functions with bounded support are dense in $X[0,\infty)$.

The space $L^1[0,\infty)+L^\infty[0,\infty)$ is defined as the set of all measurable functions f on $[0,\infty)$ endowed with the norm

$$\|f\| = \inf\{\|g\|_1 + \|h\|_\infty;\ f = g + h\}.$$

$L^1[0,\infty)+L^\infty[0,\infty)$ is clearly a r.i. function space on $[0,\infty)$. again it is easily verified that condition (iii′) is equivalent to the fact that $\chi_{[0,1]}$ has norm one in $X[0,\infty)$. In general, the dual X^* of a r.i. function space X need not be a space of functions but, one can use instead the subspace X' of X^* consisting of the integrals; i.e., of those $x^* \in X^*$ for which there exists a measurable function φ so that

$$x^*(f) = \int_I f\varphi d\mu,$$

for every $f \in X$. The condition imposed on X guarantee that X' is a norming subspace of X^* which satisfies (i), (ii) and (iii) respectively, (iii′). Therefore, the closure of the simple functions with bounded support in X' is a genuine r.i. function space, according to the above definition.

Since we need to talk about the Boyd interpolation theorem we will recall the definition of the indices introduced by D. Boyd in [Boy69]. If X is a r.i. function space on $[0,1]$, define the dilation mapping $D_s,\ 0 < s < \infty$, by the formula

$$(D_s f)(t) = f(st),\ t \in [0,1], \text{ and } f \in X.$$

In order to make this definition meaningful, the function f is extended to $[0,\infty]$ by $f(u) = 0$ for $u > 1$. Define now the indices

$$\alpha_X = \inf_{0<s<1}\left(\frac{-Log\|D_s\|}{Log\, s}\right);\ \beta_X = \sup_{1<s<\infty}\left(\frac{-Log\|D_s\|}{Log\, s}\right).$$

The numbers $\alpha_X,\ \beta_X$ belong to the closed interval $[0,1]$ and are called the Boyd indices of X.

In [LT79], if $I = [0, \infty)$, then for a measurable function f on I, define

$$E_s(f)(t) = f(\frac{t}{s}), \quad 0 < s < \infty, \ 0 \leqslant t < \infty$$

and if $I = [0, 1]$, then for a measurable function f on I and $0 < s < \infty$ define

$$E_s(f)(t) = \begin{cases} f(\frac{t}{s}) & \text{if } t \leqslant \min(1, s) \\ 0 & \text{if } s < t \leqslant 1, \ (\textit{in case } s < 1). \end{cases}$$

Then the indices p_X and q_X are defined by:

$$p_X = \lim_{s \to \infty} \frac{\log s}{\log \|E_s\|} = \sup_{s > 1} \frac{\log s}{\log \|E_s\|}$$

$$q_X = \lim_{s \to 0^+} \frac{\log s}{\log \|E_s\|} = \inf_{0 < s < 1} \frac{\log s}{\log \|E_s\|}$$

In Boyd's paper [Boy69] and several other places in the literature the indices of the r.i. function space X are taken to be the reciprocals of the p_X and q_X. (i.e., $\alpha_X = \frac{1}{p_X}$ and $\beta_X = \frac{1}{q_X}$).

Proposition 2.1.1. ***[LT79, Proposition 2.b.2].*** *Let X be a r.i. function space. Then*
(i) $1 \leqslant p_X \leqslant q_X \leqslant \infty$
(ii) $\frac{1}{p_X} + \frac{1}{q_{X'}} = 1$, $\frac{1}{q_X} + \frac{1}{p_{X'}} = 1$.

Proposition 2.1.2. ***[LT79, Proposition 2.b.3].*** *Let X be a r.i. function space on an interval I which is either $[0,1]$ or $[0,\infty)$. Then, for every $1 \leqslant p < p_X$ and $q_X < q \leqslant \infty$, we have*

$$L^p(I) \cap L^q(I) \subset X \subset L^p(I) + L^q(I),$$

with the inclusion maps being continuous.

If $I = [0,1]$ then $L^p(I) \cap L^q(I) = L^q(I)$ and $L^p(I) + L^q(I) = L^p(I)$. Proposition (2.1.2) can be used to describe the behaviour of the Rademacher functions in a r.i. function space $X[0,1]$. For instant, if $q_X < \infty$ and $q > q_X$ then, by Proposition (2.1.2), there exists a constant $K < \infty$ such that $\|f\|_X \leqslant K\|f\|_q$ for all $f \in L^q[0,1]$. Hence, by Khintchine's inequality, we have

$$A_1(\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} \leqslant \|\sum_{i=1}^{n} a_i r_i\|_1 \leqslant \|\sum_{i=1}^{n} a_i r_i\|_X \leqslant K\|\sum_{i=1}^{n} a_i r_i\|_q \leqslant K B_q(\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} \tag{2.1.1}$$

for every choice of $\{a_i\}_{i=0}^n$. This proves that in any r.i. function space $X[0,1]$ with $q_X < \infty$ the Rademacher functions are equivalent to the unit vector basis in ℓ^2. A Banach lattice X is said to be p-convex, $1 < p < \infty$ with p-convexity constant $\leqslant M$ (q-concave with q-concavity constant $\leqslant M$) provided that:

$$\|(\sum_{i=1}^{n} |x_i|^p)^{1/p}\| \leqslant M(\sum_{i=1}^{n} \|x_i\|^p)^{1/p}$$

$$\|(\sum_{i=1}^{n} |x_i|^q)^{1/q}\| \geqslant M(\sum_{i=1}^{n} \|x_i\|^q)^{1/q},$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X, (see [JMST79, p. 13] or [LT79, section 1.d]). It was shown in [JMST79, p. 206-208] that when the $X[0,1]$ is p-convex (q-concave), then $\alpha_X \leqslant 1/p$ ($\beta_X \geqslant 1/q$).

We recall that a Banach lattice is super-reflexive if and only if it is p-convex and q-concave for some $p > 1$ and $q < \infty$. Therefore, the Boyd indices of a super-reflexive r.i. function space X satisfy $0 < \beta_X \leqslant \alpha_X < 1$. The converse however is not true, and there are non-reflexive r.i. function spaces X on $[0,1]$ for which $\beta_X = \alpha_X = 1/2$, (see [JMST79, section 8]).

Proposition 2.1.3. ***[LT79, Proposition 1.d.4].*** *Let X be a Banach lattice and $1 \leqslant p,\ q \leqslant \infty$ so that $\frac{1}{p} + \frac{1}{q} = 1$. The space X is p-convex (concave) if and only if X^* is q-concave (convex).*

2.2 Boyd interpolation theorem

We turn now to the interpolation theorem which motivated the definition of the Boyd indices and which will play an important role in the sequel. We have first to define the notion of the operator of weak type (p,q). The perhaps most natural way to introduce this notion is by using $L_{p,q}$.

Definition 2.2.1. [LT79, Definition 2.b.8]. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a measure space. For $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, $L_{p,q}(\Omega, \mathfrak{F}, \mathbb{P})$ is the space of all locally integrable real valued functions f on Ω for which

$$\|f\|_{p,q} = [\tfrac{q}{p} \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \tfrac{dt}{t}]^{\frac{1}{q}} < \infty$$

For $1 \leqslant p \leqslant \infty$, $L_{p,\infty}(\Omega, \mathfrak{F}, \mathbb{P})$ is the space of all locally integrable real valued functions f on Ω for which

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$$

Note that, for $p = q$, $L_{p,q}$ coincides with L^p (with the same norm).

Proposition 2.2.2. ***[LT79, Proposition 2.b.9].*** *Let $1 \leqslant p < \infty$ and $1 \leqslant q_1 < q_2 \leqslant \infty$. Then*

$$L_{p,q_1}(\Omega, \Sigma, \nu) \subset L_{p,q_2}(\Omega, \Sigma, \nu)$$

and moreover, for every $f \in L_{p,q_1}(\Omega, \Sigma, \nu)$,

$$\|f\|_{p,q_2} \leqslant \|f\|_{p,q_1}.$$

There are also the inclusion relations between the $L_{p,q}$ spaces as p varies. If (Ω, Σ, ν) is a probability measure space then, for every $r < p < s$ and every q,

$$L_{s,\infty}(\Omega, \Sigma, \nu) \subset L_{p,q}(\Omega, \Sigma, \nu) \subset L_{r,1}(\Omega, \Sigma, \nu)$$

Definition 2.2.3. [LT79, Definition 2.b.10]. Let $(\Omega_i, \Sigma_i, \nu_i)$, $i = 1, 2$, be two measure spaces. Let $1 \leqslant p_1 \leqslant \infty$ and let T be a map defined on a subset of $L^{p_1}(\Omega_1)$ which takes values in the space of all measurable functions in Ω_2.

(i) The map T is said to be of *strong type* (p_1, p_2), for some $1 \leqslant p_2 \leqslant \infty$, if there is a constant M so that

$$\|Tf\|_{p_2} \leqslant M\|f\|_{p_1},$$

for every f in the domain of the definition of T.

(ii) The map T is called *weak type* (p_1, p_2) , for some $1 \leqslant p_2 \leqslant \infty$, if there is a constant M so that

$$\|Tf\|_{p_2,\infty} \leqslant M\|f\|_{p_1,1}$$

for every f in the domain if definition of T with the convention that if $p_1 = \infty$ we have to replace $\|f\|_{\infty,1}$ above, which is not defined, by $\|f\|_{\infty,\infty} = \|f\|_\infty$.

It is clear from Proposition (2.2.2) that $\|f\|_{p_1} = \|f\|_{p_1,p_1} \leqslant \|f\|_{p_1,1}$. Now, we will state the the interpolation theorem of Boyd, (see [Boy69] or [LT79, Theorem 2.b.11]).

Theorem 2.2.4. *Let I be either $[0,1]$ or $[0,\infty)$, let $1 \leqslant p < q \leqslant \infty$ and let T be a linear operator mapping $L_{p,1}(I) + L_{q,1}(I)$ into the space of measurable functions on I. Assume that T is of weak types (p,p) and (q,q) (with respect to the Lebesgue measure on I). Then, for every r.i. function space X on I, so that $p < p_X$ and $q_X < q$, T maps X into itself and is bounded on X.*

We shall use later in this chapter a weaker version of Boyd's interpolation theorem [JMST79, p. 208]:
Let $X[0,1]$ be a rearrangement invariant function space, p, q such that $0 < \frac{1}{q} < \beta_X \leqslant \alpha_X < \frac{1}{p} < 1$, and $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A linear transformation L, which is bounded from $L^p(\Omega, \mathfrak{F}, \mathbb{P})$ to itself and from $L^q(\Omega, \mathfrak{F}, \mathbb{P})$ to itself defines a bounded operator from $X(\Omega, \mathfrak{F}, \mathbb{P})$ into itself.

2.3 Orlicz spaces

The most commonly used r.i. function spaces on $[0,1]$ besides the L^p spaces, $1 \leqslant p \leqslant \infty$, are the Orlicz function spaces. We refer the interested reader to [LT77], [LT79] and [RR02] for more about the Orlicz spaces.

Definition 2.3.1. The *Orlicz function* Φ is a non negative convex function on $[0,\infty)$ such that $\Phi(0) = 0$ and $lim_{t\to\infty}\Phi(t) = \infty$. If $\Phi(t) = 0$ for some $t > 0$, Φ is said to be a degenerate Orlicz function.

Definition 2.3.2. Let $(\Omega, \mathcal{F}, \mu)$ be a separable measure space and Φ be an Orlicz function. The *Orlicz function space* $L^\Phi(\Omega, \mathcal{F}, \mu)$ is the space of all (equivalence classes of) $\mathcal{F}$-measurable functions f so that

$$\textstyle\int_\Omega \Phi\left(\frac{|f|}{\rho}\right) < \infty$$

for some $\rho > 0$. The norm is defined by

$$\|f\|_\Phi = \inf\{\rho > 0; \textstyle\int_\Omega \Phi\left(\frac{|f|}{\rho}\right) \leqslant 1\}$$

In addition, we require the normalization $\Phi(1) = 1$, in order that $\|\chi_{[0,1]}\|_\Phi = 1$ in $L^\Phi[0,1]$ and $L^\Phi[0,\infty)$.

Definition 2.3.3. For any Orlicz function Φ we associate the space ℓ^Φ of all sequences of scalars $x = (x_1, x_2, \cdots)$ such that $\sum_{n=1}^\infty \Phi\left(\frac{|x_n|}{\rho}\right) < \infty$ for some $\rho > 0$. The space ℓ^Φ equipped with the norm

$$\|x\|_\Phi = \inf\{\rho > 0;\ \sum_{n=1}^\infty \Phi\left(\frac{|x_n|}{\rho}\right) \leqslant 1\}$$

is a Banach space usually called an *Orlicz sequence space*. The subspace h_Φ of ℓ^Φ consists of those sequences $x = (x_1, x_2, \cdots)$ such that $\sum_{n=1}^\infty \Phi\left(\frac{|x_n|}{\rho}\right) < \infty$ for every $\rho > 0$. Moreover, the unit vectors $\{e_i\}_{i=1}^\infty$ is a symmetric basis of h_Φ.

If Φ is a degenerate Orlicz function then $\ell^\Phi \cong \ell^\infty$ and $h_\Phi \cong c_0$. Since this case is not interesting in the present context we assume from now on that all the Orlicz functions considered in the sequel are non-degenerate.

Definition 2.3.4. The *weighted Orlicz sequence spaces* $\ell^\Phi(\omega)$ determined by the Orlicz function Φ and a sequence $\omega = (\omega_n)$ of positive scalars are defined by

$$\ell^\Phi(\omega) = \{x = (x_n);\ \sum_{n=1}^\infty \omega_n \Phi\left(\frac{|x_n|}{\rho}\right) < \infty\ for\ some\ \rho > 0\}$$

with norm

$$\|x\|_{\Phi,\omega} = \inf\{\rho > 0;\ \sum_{n=1}^\infty \omega_n \Phi\left(\frac{|x_n|}{\rho}\right) \leqslant 1\}$$

In particular, if $\omega_n = 1$ for all n, then $\ell^\Phi(\omega) = \ell^\Phi$.

Definition 2.3.5. Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of Orlicz functions. The space $\ell^{(\Phi_n)}$ is the Banach space of all sequences $x = (x_n)$ with $\sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) < \infty$ for some $\rho > 0$, equipped with the norm

$$\|x\|_{(\Phi_n)} = \inf\{\rho > 0;\ \sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) \leqslant 1\}$$

The space $\ell^{(\Phi_n)}$ is called a *modular sequence space*.
An important subspace of $\ell^{(\Phi_n)}$ is $h_{(\Phi_n)}$ which consists of those sequences $x = (x_1, x_2, \cdots)$ such that $\sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) < \infty$ for every $\rho > 0$.

Definition 2.3.6. An Orlicz function Φ is said to obey the Δ_2-*condition* for large t (for small t, or for all t), written often as $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2(0)$, $\Phi \in \Delta_2$), if there exist constants $t_0 > 0, K > 2$ such that $\Phi(2t) \leqslant K\Phi(t)$ for $t \geqslant t_0$ (for $0 < t < t_0$, or for all $t > 0$); and $\Phi \in \nabla_2(\infty)$ ($\Phi \in \nabla_2(0)$, $\Phi \in \nabla_2$), if there exist constants $t_0 > 0, c > 1$ such that $\Phi(t) \leqslant \frac{1}{2c}\Phi(ct)$ for $t \geqslant t_0$ (for $0 < t < t_0$, or for all $t > 0$).

In fact, one can see that the Orlicz function $\Phi \in \Delta(\infty) \cap \nabla(\infty)$ if and only if there exist $\lambda_0 > 0$ and $t_0 > 0$ such that for every $\lambda > \lambda_0$ there exist positive constants c_λ, C_λ such that

$$c_\lambda \Phi(t) \leqslant \Phi(\lambda t) \leqslant C_\lambda \Phi(t), \quad \forall t \geqslant t_0.$$

If $\Phi \in \Delta_2(0)$, then $\ell^\Phi = h_\Phi$, hence, it is separable (see [LT77, Proposition 4.a.4]). In addition, the Orlicz function space $L^\Phi[0,1]$ is separable if and only if $\Phi \in \Delta_2(\infty)$. We

recall that non-negative functions Φ_1 and Φ_2 on $[0,\infty)$ are said to be equivalent at ∞ (respectively, at 0) provided that there are $0 < t_0 < \infty$, and $0 < A, B, a, b$, such that $A\Phi_2(at) \leqslant \Phi_1(t) \leqslant B\Phi(bt)$ for all $t > t_0$, (respectively, $t < t_0$).
Observe that, up to an equivalent renorming, the space $L^{\Phi}[0,1]$ is determined by the behavior of $\Phi(t)$ for $t \geqslant 1$; i.e., if an Orlicz function Φ_1 is equivalent to Φ at ∞, then $L^{\Phi_1}[0,1] = L^{\Phi}[0,1]$ (with equivalent norm).

Sequence of Orlicz functions $\{\Phi_n\}_{n=1}^{\infty}$ is said to satisfy the uniform Δ_2-condition at zero if there exists a number $p > 1$ and an integer n_0 such that, for all $t \in (0,1)$ and $n \geqslant n_0$, we have $t\Phi^{\backprime}_n(t)/\Phi_n(t) \leqslant p$, where $\Phi^{\backprime}$ is a right continuous function of Φ. Consequently, the Orlicz sequence $\{\Phi_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition if and only if $\ell^{(\Phi_n)} = h_{(\Phi_n)}$ (i.e, the unit vectors form an unconditional basis of $\ell^{(\Phi_n)}$), see [LT77, Proposition 4.d.3].

Proposition 2.3.7. *Let Φ be an Orlicz function and $\Phi^{\backprime}$ be the right continuous derivative of Φ such that $\Phi^{\backprime}(0) = 0$ and $\lim_{t\to\infty} \Phi^{\backprime}(t) = \infty$. The following are equivalent:*

1. *The Orlicz function Φ satisfies the Δ_2-condition for large $t > 0$ (for all $t > 0$).*
2. *For large t (for all $t > 0$) and for $Q > 0$, then there exists $C > 1$ such that*

$$\Phi(Qt) \leqslant C\Phi(t)$$

3. *For large t (for all $t > 0$)*

$$\limsup_{t\to\infty} \frac{t\Phi^{\backprime}(t)}{\Phi(t)} < +\infty \quad \left(\sup_{t>0} \frac{t\Phi^{\backprime}(t)}{\Phi(t)} < +\infty\right)$$

Proof. (1)$\Rightarrow$ (2) Let Φ satisfies the Δ_2-condition for large t (all $t > 0$). Then there exists a constant $K > 2$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{2.3.1}$$

for large t (all $t > 0$). Let $Q > 1$ (for $Q \in [0,1]$ the inequality holds by the convexity of Φ). Consider $n \in \mathbb{N}$ such that $Q < 2^n$. Therefore, for large t (all $t > 0$), we have

$$\Phi(Qt) \leq \Phi(2^n t) \leq K^n \Phi(t)$$

(2)$\Rightarrow$(1) Let $Q = 2$, then (1) directly holds.
(1)$\Rightarrow$(3) Let $\Phi(2t) \leq K\Phi(t)$. Then

$$K\Phi(t) \geq \Phi(2t) = \int_0^{2t} \Phi^{\backprime}(u)du > \Phi(2t) - \Phi(t) = \int_t^{2t} \Phi^{\backprime}(u)du > t\Phi^{\backprime}(t)$$

(3)$\Rightarrow$(1) Let $\frac{t\Phi^{\backprime}(t)}{\Phi(t)} < \alpha$ for large t (all $t > 0$). Since $t\Phi^{\backprime}(t)$ is always $> \Phi(t)$, then $\alpha > 1$. Hence

$$\int_t^{2t} \frac{\Phi^{\backprime}(u)}{\Phi(u)}du < \alpha \int_t^{2t} \frac{du}{u} = \ln 2^{\alpha}$$

or, equivalently, that $\Phi(2t) < 2^{\alpha}\Phi(t)$, for large t (all $t > 0$). $\square$

Proposition 2.3.8. *The Orlicz function Φ satisfies the Δ_2-condition for all $t > 0$ if and only if there exists $1 < q < \infty$ such that $\Phi(\rho t) \leq \rho^q \Phi(t)$ for all $\rho > 1$.*

Proof. Let Φ satisfies the Δ_2-condition for $t > 0$, equivalently, there exists $K > 2$ such that

$$\Phi(Qt) \leq K\Phi(t) \tag{2.3.2}$$

for all $t > 0$ and $Q > 1$. Moreover, this is true if and only if $q = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}$ satisfies $1 < q \leq K - 1 < \infty$ by the proof of Proposition (2.3.7). Therefore,

$$t\Phi'(t) \leq q\Phi(t)$$

for all $t > 0$. Then for $\rho > 1$ we have

$$\int_t^{\rho t} \frac{\Phi'(u)}{\Phi(u)} du < q \int_t^{\rho t} \frac{du}{u} = \ln \rho^q$$

or, equivalently, that $\Phi(2t) < \rho^q \Phi(t)$ for all $t > 0$ and $\rho > 1$.
The converse is trivial by taking $\rho = 2$. □

Corollary 2.3.9. *[RR91, Corollary 12, p. 113]. Let (Ω, Σ, μ) be the finite measure space and Φ be an Orlicz function. Then $L^\Phi(\mu)$ is reflexive if and only if $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$.*

Definition 2.3.10. [RR02, Definition II-1, p. 50]. For a Banach space X, the parameter $J(X)$ is termed a nonsquare constant, where

$$J(X) = \sup\{min(\|x - y\|, \|x + y\|); x, y \in S_X\}.$$

A Banach space Y is finitely representable in X if for any $a > 1$ and each finite dimensional subspace $Y_1 \subset Y$, there is an isomorphism T from Y_1 into X such that for all $x \in y_1$ one has:

$$\frac{1}{a}\|x\| \leqslant \|Tx\| \leqslant a\|x\|.$$

A Banach space X is super-reflexive if no nonreflexive Banach space is finitely representable in X.

Corollary 2.3.11. *[RR02, Corollary II-8, p. 53]. Let $\widetilde{J}(X)$ be the infima of nonsquare constants of equivalent norms of X. Then the following assertions are equivalent:*
(i) X is super-reflexive,
(ii) $\widetilde{J}(X) < 2$.

Theorem 2.3.12. *[RR02, Theorem II.3.2, p.61]. For each Orlicz function $\Phi \in \Delta_2(\infty) \cap \nabla_2(\infty)$ one has $J(L^\Phi[0, 1]) < 2$.*

In view of the above results, it is clear that the Orlicz function space $L^\Phi[0, 1]$ is super-reflexive if and only if it is reflexive.
The space $L^\Phi[0, \infty)$ is p-convex with constant one if and only if $\Phi(t^{\frac{1}{p}})$ is a convex function. In general, $L^\Phi[0, \infty)$ (respectively, $L^\Phi[0, 1]$) is p-convex if and only if there is a constant M_1 such that, whenever $0 < s \leq t < \infty$, $\frac{\Phi(s)}{s^p} \leq M_1 \frac{\Phi(t)}{t^p}$. This is the same as saying that there is a function Φ_1 equivalent to Φ (respectively, equivalent at ∞) such that $\Phi_1(t^{\frac{1}{p}})$ is convex function. A similar argument shows that L^Φ is q-concave if and only if there is a constant M_2 such that, whenever $0 < s \leq t < \infty$, $\frac{\Phi(t)}{t^q} \leq M_2 \frac{\Phi(s)}{s^q}$. This is exactly the same as saying that there is a function Φ_2 equivalent to Φ such that $\Phi_2(t^{\frac{1}{q}})$ is concave, (see [JMST79, p.168]).

Proposition 2.3.13. *The Orlicz space $L^{\Phi}[0,1]$ is q-concave if and only if Φ satisfies the Δ_2-condition.*

Proof. By Proposition (2.3.7), the Orlicz function Φ satisfies the Δ_2-condition for all $t > 0$ if and only if $q = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}$ satisfies $1 < q < \infty$. Moreover, this is equivalent to $t\Phi'(t) < q\Phi(t)$ for all $t > 0$ and this holds if and only if $\frac{\Phi(t)}{t^q}$ is non-increasing for all $t > 0$. Therefore, by previous argument this is equivalent to say that L^{Φ} is q-concave. □

The following result of the W. Boyd [Boy71] gives another description of the Boyd indices for the Orlicz function space.

Proposition 2.3.14. ***[Boy71], [LT79, Proposition 2.b.5].*** *Let $L^{\Phi}[0,1]$ be an Orlicz function space. Then*

$$p_{\Phi} = \sup\Big\{p;\ \inf_{\lambda,t\geqslant 1} \frac{\Phi(\lambda t)}{\Phi(\lambda)t^p} > 0\Big\}$$

$$q_{\Phi} = \inf\Big\{q;\ \sup_{\lambda,t\geqslant 1} \frac{\Phi(\lambda t)}{\Phi(\lambda)t^q} < \infty\Big\}$$

and so the Boyd indices are $\alpha_{\Phi} = \frac{1}{p_{\Phi}}$ and $\beta_{\Phi} = \frac{1}{q_{\Phi}}$.

We recall that, for Orlicz function space $L^{\Phi}[0,1]$, the conditions $1 < p_{\Phi}$ and $q_{\Phi} < \infty$ (i.e., $0 < \beta_{\Phi} \leqslant \alpha_{\Phi} < 1$) are equivalent to the reflexivity of $L^{\Phi}[0,1]$.

2.4 Property (M)

Many results were proved about M-ideals and property (M) of Banach spaces, (see e.g. [Kal93] and [KW95]). We recall here the definition of the property (M) and some required results.

Definition 2.4.1. A Banach space X has property (M) if whenever $u, v \in X$ with $\|u\| = \|v\|$ and (x_n) is weakly null sequence in X then,

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|.$$

The technique of renorming which is presented in this section was introduced by N. J. Kalton in [Kal93].
Let $\mathcal{N}$ be the family of all norms N on $\mathbb{R}^2$ such that

$$N(1,0) = 1 \textit{ and } \forall s,t \in \mathbb{R},\ N(s,t) = N(|s|,|t|),$$

If $N_1, \ldots, N_k \in \mathcal{N}$, we define a norm $N_1 * \ldots * N_k$ on $\mathbb{R}^{k+1} = \{(x_i)_{i=0}^k\}$ by

$$N_1 * N_2(x) = N_2(N_1(x_0, x_1), x_2)$$

and then inductively by the rule

$$N_1 * \ldots * N_k(x) = N_k(N_1 * \ldots * N_{k-1}(x_0, \ldots, x_{k-1}), x_k).$$

Next, suppose $(N_k)_{k=1}^{\infty}$ is any sequence of norms in $\mathcal{N}$. We define the sequence space $\widetilde{\Lambda}(N_k)$ to be the space of all sequences $(x_i)_{i=1}^{\infty}$ such that

$$\|x\|_{\widetilde{\Lambda}(N_k)} = \sup_{1\leqslant k<\infty} (N_1 * \ldots * N_k(x_0, \ldots, x_k)) < \infty.$$

Then $\widetilde{\Lambda}(N_k)$ is a Banach space. We define $\Lambda(N_k)$ to be the closed linear span of the basis vectors $(e_k)_{k=0}^{\infty}$ in $\Lambda(N_k)$. It is easy to verify the next result using a gliding hump argument.

Proposition 2.4.2. ***([Kal93, Proposition 3.2])****. The space $\Lambda(N_k)$ has property (M).*

We now turn to identifying the spaces $\Lambda(N_k)$. Let us define $\Phi_k(t) = N_k(1, t) - 1$. For convenience let $F_0(t) = |t|$.

Proposition 2.4.3. ***([Kal93, Proposition 3.3])****. The real sequence $x = (x_k)_{k=0}^{\infty}$ belongs to $\widetilde{\Lambda}(N_k)$ if and only if for some $\alpha > 0$, $\sum_{k=0}^{\infty} \Phi_k(\alpha x_k) < \infty$.*

Proof. First, suppose $x \in \widetilde{\Lambda}(N_k)$ and $\|x\|_{\widetilde{\Lambda}(N_k)} \leqslant 1$. Let $h \geqslant 0$ be the first index such that $|x_k| > 0$. Then for $k \geqslant h + 1$,

$$N_1 * \ldots * N_k(x_0, \ldots, x_k) \geqslant (1 + \Phi_k(x_k))N_1 * \ldots * N_{k-1}(x_0, \ldots, x_{k-1})$$

so that

$$\prod_{k=h+1}^{\infty} (1 + \Phi_k(x_k)) < \infty$$

whence $\sum_{k=0}^{\infty} \Phi_k(x_k) < \infty$. This quickly establishes one direction.
For the other direction assume $x \notin \widetilde{\Lambda}(N_k)$. Then there exists h such that

$$N_1 * \ldots * N_h(x_0, \ldots, x_h) > 1.$$

A similar argument to the above shows that

$$\prod_{k=h+1}^{\infty} (1 + \Phi_k(x_k)) = \infty$$

This completes the proof. □

In Definition (2.3.5) we recall the definition of the modular space $\ell^{(\Phi_n)}$ and the closed linear span on the unit vectors (e_n) is denoted by $h_{(\Phi_n)}$, where (Φ_n) is a sequence of Orlicz functions.

Proposition 2.4.4. ***[Kal93, Proposition 3.4]****. Suppose for $k \geqslant 1$, $N_k \in \mathcal{N}$ and $\Phi_k(t) = N_k(1, t) - 1$. Then $\Lambda(N_k)$ is canonically isomorphic to $h_{(\Phi_k)_{k=0}^{\infty}}$ where $\Phi_0(t) = |t|$. The induced norms are equivalent.*

Proposition 2.4.5. ***[Kal93, Proposition 4.1]****. A modular sequence space $X = h_{(\Phi_k)}$ can be equivalently normed to have property (M).*

Proof. We may assume that $\Phi_k(1) = 1$ for all k. Note that the right derivative of each Φ_k at $\frac{1}{2}$ is at most 2 (using the fact that $\Phi_k(2t) > t\Phi_k^{`}(t)$ for all $t > 0$). It suffices to replace $(\Phi_k)_{k=1}^{\infty}$ by an equivalent sequence $(\varphi_k)_{k=1}^{\infty}$ of the form $\varphi_k(t) = N_k(1, t) - 1$ for suitable $N_k \in \mathcal{N}$. We will choose each φ_k to be convex and satisfy the condition that $t^{-1}(\varphi_k(t) + 1)$ is monotone decreasing for $t > 0$. This is achieved by putting

$$\varphi_k(t) = \begin{cases} \Phi_k(t) & \text{if, } 0 \leqslant t \leqslant 1/2 \\ \Phi_k(\frac{1}{2}) + 2t - 1 & \text{if, } t > 1/2. \end{cases}$$

Now let

$$N_k(\alpha,\beta)=\begin{cases}|\alpha|(1+\varphi_k(\frac{|\beta|}{|\alpha|})) & \text{if, } \alpha\neq 0,\\ 2|\beta| & \text{if, } \alpha=0.\end{cases}$$

Then N_k is an absolute norm $\mathbb{R}^2$.
Suppose $s_1,s_2>0$ and $t_1,t_2>0$. Then

$$\begin{aligned}N_k(s_1+s_2,t_1+t_2)&=(s_1+s_2)+(s_1+s_2)\varphi_k(\frac{t_1+t_2}{s_1+s_2})\\&=(s_1+s_2)+(s_1+s_2)\varphi_k(\frac{s_1}{s_1+s_2}\frac{t_1}{s_2}+\frac{s_2}{s_1+s_2}\frac{t_2}{s_2})\\&\leqslant s_1+s_2+s_1\varphi_k(\frac{t_1}{s_1})+s_2\varphi_k(\frac{t_1}{s_1})\\&=N_k(s_1,t_1)+N_k(s_2,t_2)\end{aligned}$$

by the convexity of φ_k.
To check the required absolute property since $N(s,t)$ is monotone increasing in t we need only to check that N is monotone in s for t fixed. If $0<s_1\leqslant s_2$ and $t>0$,

$$\frac{s_1}{t}\left(1+\varphi_k(\frac{t}{s_1})\right)\leqslant\frac{s_2}{t}\left(1+\varphi_k(\frac{t}{s_2})\right)$$

whence it follows that $N_k(s_1,t)\leqslant N_k(s_2,t)$; again the other case when one or more of s_1,t vanish is treated as a limiting case. □

For the weighted sequence space $\ell^{\overline{\Phi}}(\omega)$ where $\omega=(\omega_n)$ is a positive real sequence, if we consider $\Phi_n=\omega_n\cdot\overline{\Phi}$, then Φ_n is a sequence of Orlicz functions such that for any sequence $x=(x_n)$ we have $\|x\|_{(\Phi_n)}=\|x\|_{\overline{\Phi},\omega}$ (i.e., $\ell^{(\Phi_n)}=\ell^{\overline{\Phi}}(\omega)$). Therefore, $\ell^{\overline{\Phi}}(\omega)$ can be equivalently normed to have property (M).

Theorem 2.4.6. ***[Kal93, Theorem 4.3].*** *Let X be a separable order-continuous nonatomic Banach lattice. If X has an equivalent norm with property (M), then X is lattice-isomorphic to L^2.*

This theorem really says that, except for the Hilbert spaces, the existence of a norm with property (M) forces the lattice to be "discrete".

2.5 Rosenthal's spaces in r.i. function spaces

The spaces X_p were Rosenthal's discovery. They play an important role in the study of the complemented subspaces of $L^p(0,1)$.

Assume $2\leqslant p<\infty$. Let $\omega=(\omega_n)$ be a sequence of real numbers with $0\leqslant\omega_n\leqslant 1$ for all n. We then consider the space $X_{p,\omega}$ consisting of all sequences of reals $x=(x_n)$, so that

$$\|x\|=\max\{(\sum|x_n|^p)^{\frac{1}{p}},(\sum\omega_n^2|x_n|^2)^{\frac{1}{2}}\}<\infty.$$

The interest of these spaces lies in their probabilistic interpretation, (see [Ros70] and [Ros72]).
In connection with X_p, W.B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in [JMST79] showed that to every r.i. function space Y on $[0,\infty)$ is associated a space U_Y with unconditional basis. The description is quite simple: choose any sequence $\{A_n\}_{n=1}^\infty$

of disjoint measurable subsets of $[0,\infty)$ such that $\lim\limits_{n\to\infty}\mu(A_n)=0$ and $\sum_{n=1}^{\infty}\mu(A_n)=+\infty$. The space U_Y is isomorphic to the closed linear span of the sequence $\{\chi_{A_n}\}_{n=1}^{\infty}$ in $Y(0,\infty)$, up to isomorphism the space U_Y does not depend on the particular choice of a sequence $\{A_n\}_{n=1}^{\infty}$. The sequence $u_n=\frac{\chi_{A_n}}{\|\chi_{A_n}\|}$, $n=1,2,\ldots$, forms an unconditional basis of the space. The space X_p is simply the space U_Y corresponding to the r.i. function space Y on $[0,\infty)$ constructed from $L^p(0,1)$.

Lemma 2.5.1. *[JMST79, Lemma 8.7]. Let Y be a r.i. function space on $[0,\infty)$. Then there is a Banach space U_Y with unconditional basis such that for every sequence $\{A_n\}_{n=1}^{\infty}$ of disjoint integrable sets $[0,\infty)$, such that $\sum_{n=1}^{\infty}\mu(A_n)=\infty$ and $\mu(A_n)\to 0$ as $n\to\infty$, the space U_Y is isomorphic to the closed linear span of the sequence $\{\chi_{A_n}\}_{n=1}^{\infty}$ in Y.*

Remark 2.5.2. [JMST79, Section 8]. Let $L^{\Phi}[0,1]$ be an Orlicz space, where Φ satisfies the Δ_2-condition at ∞ (which is equivalent to $0<\beta_\Phi$ in this case). The space $L^{\overline{\Phi}}[0,\infty)$ is isomorphic to $L^{\Phi}[0,1]$ if $\overline{\Phi}$ is equivalent to t^2 at 0 and to Φ at ∞, for example: $\overline{\Phi}(t)=t^2\chi_{[0,1)}(t)+(2\Phi(t)-1)\chi_{[1,\infty)}(t)$. In addition, we have the following results:

1. In Lemma (2.5.1), the space $U_{L^{\overline{\Phi}}[0,\infty)}=U_{\overline{\Phi}}$ is isomorphic to the span in $L^{\Phi}[0,1]$ of any sequence $\{T_n\}_{n=1}^{\infty}$ of independent random variables taking only values 0, ± 1 such that $\mathbb{E}T_n=0$ for every n, $\lim\limits_{n\to\infty}\mathbb{E}T_n^2=0$ and $\sum_{n=1}^{\infty}\mathbb{E}T_n^2=+\infty$.
2. Furthermore, it is clear that for every sequence $\{A_n\}_{n=1}^{\infty}$ of disjoint integrable sets $[0,\infty)$, such that $\sum_{n=1}^{\infty}A_n=\infty$ and $\mu(A_n)\to 0$ as $n\to\infty$, the space $[\chi_{A_n}]$ is isometrically isomorphic to the weighted Orlicz sequence spaces $\ell^{\overline{\Phi}}(\omega)$ such that $\omega=(\omega_n)_{n=1}^{\infty}$ with $\omega_n=\mu(A_n)$.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^{\Phi}[0,1]$. We denote by $(\overline{f}_n)_{n=1}^{\infty}$ the sequence of $L^{\overline{\Phi}}[0,\infty)$ defined by $\overline{f}_n(t)=f_n(t-n+1)\chi_{[n-1,n)}(t)$ where $t>0$. Thus, the sequence $(\overline{f}_n)_{n=1}^{\infty}$ is a disjointification of the sequence $(f_n)_{n=1}^{\infty}$. By [JS89, Theorem 1] we can find a constant $C\geq 1$, which depends only on Φ, such that if $(f_n)_{n=1}^{\infty}$ be a sequence of independent mean zero random variables, then:

$$C^{-1}\|\sum_{n=1}^{m}\overline{f}_n\|_{L^{\overline{\Phi}}[0,\infty)}\leq\|\sum_{n=1}^{m}f_n\|_{L^{\Phi}[0,1]}\leq C\|\sum_{n=1}^{m}\overline{f}_n\|_{L^{\overline{\Phi}}[0,\infty)}. \tag{2.5.1}$$

for more about this inequality and its applications see [JS89], [Rui97] and [ASW11].

Proposition 2.5.3. *Let $L^{\Phi}[0,1]$ be a reflexive Orlicz space and $(f_n)_{n=1}^{\infty}$ be a sequence of independent mean zero random variables in $L^{\Phi}[0,1]$. Then $(f_n)_{n=1}^{\infty}$ is equivalent to the sequence of unit vectors $(e_n)_{n=1}^{\infty}$ of the modular sequence space $\ell^{(\varphi_n)}$ for some $(\varphi_n)_{n=1}^{\infty}$.*

Proof. Let $(\overline{f}_n)_{n=1}^{\infty}$ be a disjointification sequence of $(f_n)_{n=1}^{\infty}$ such that $\overline{f}_n(t)=f_n(t-n+1)\chi_{[n-1,n)}(t)$ where $t>0$. Moreover, define the Orlicz functions $\varphi_n(s)=\int_{n-1}^{n}\overline{\Phi}(s|f_n(t-n+1)|)dt$, $n\in\mathbb{N}$. Now, assume $(e_n)_{n=1}^{\infty}$ be the sequence of the unit vectors in $\ell^{(\varphi_n)}$. Furthermore, let $(a_n)_{n=1}^{m}$ be real numbers. Then for any $\lambda>0$, we have

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n\left(\frac{|a_n|}{\lambda}\right)&=\sum_{n=1}^{m}\int_{n-1}^{n}\overline{\Phi}\left(\frac{|a_n|}{\lambda}|f_n(t-n+1)|\right)dt\\&=\int_0^{\infty}\overline{\Phi}\left(\frac{|\sum_{n=1}^{m}a_n\overline{f}_n(t)|}{\lambda}\right)dt\end{aligned}$$

Therefore, $\|\sum_{n=1}^{m}a_ne_n\|_{(\varphi_n)}=\|\sum_{n=1}^{m}a_n\overline{f}_n\|_{L^{\overline{\Phi}}[0,\infty)}$. Thus, inequality (2.5.1) implies that the sequence $(f_n)_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$ in $\ell^{(\varphi_n)}$. □

Now, we need the following results about subspaces of $X[0,1]$ that are isomorphic to $X[0,1]$.

Theorem 2.5.4. ***[JMST79, Theorem 9.1].*** *Let $X[0,1]$ be a r.i. function space such that $X[0,1]$ is q-concave for some $q < \infty$, the index $\alpha_X < 1$ and the Haar system in $X[0,1]$ is not equivalent to a sequence of disjoint functions in $X[0,1]$. Then any subspace of $X[0,1]$ which is isomorphic to $X[0,1]$ contains a further subspace which is complemented in $X[0,1]$ and isomorphic to $X[0,1]$. In particular , the theorem holds for $X[0,1] = L^p[0,1]$, $1 < p < \infty$, and more generally, for every reflexive Orlicz function space $L^{\Phi}[0,1]$.*

The proof of the following corollary in the case of L^p, $1 < p < \infty$, is a straightforward consequence of Theorem (2.5.4) and the Pełczyński's decomposition method; it works also for reflexive Orlicz function spaces $L^{\Phi}[0,1]$, since the Haar basis in the reflexive Orlicz function space $L^{\Phi}[0,1]$ cannot be equivalent to the unit vector basis of a modular sequence space, unless $L^{\Phi}[0,1]$ is the Hilbert space.

Corollary 2.5.5. ***[JMST79, Corollary 9.2].*** *Let $X[0,1]$ be a r.i. function space satisfying the assumptions of Theorem (2.5.4). If Y is a complemented subspace of $X[0,1]$ which contains an isomorphic copy of $X[0,1]$, then Y is isomorphic to $X[0,1]$.*

2.6 Some basic probabilistic facts

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and $B \in \mathfrak{F}$ with $0 < \mathbb{P}(B) < 1$. Then, for any $A \in \mathcal{F}$, the conditional probability of A given B is defined to be $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$. If one writes $\mathbb{P}(A \cap B)$ as $\mathbb{E}[\chi_A \chi_B]$, then this formula extends from random variables of the form χ_A to any positive random variables X. What results is the conditional expectation of X given B: namely, $\mathbb{E}[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X d\mathbb{P}$. Similarly, one may define the conditional expectation of X given $\Omega \backslash B$: $\mathbb{E}[X|\Omega \backslash B] = \frac{1}{\mathbb{P}(\Omega \backslash B)} \int_{\Omega \backslash B} X d\mathbb{P}$. In this way, to each non-negative random variable X, one associates two values: $\mathbb{E}[X|B]$ and $\mathbb{E}[X|\Omega \setminus B]$. With these two values, one may define a random variable Y on the probability space $(\Omega, \{\emptyset, B, \Omega \setminus B, \Omega\}, \mathbb{P})$ by the formula

$$Y = \mathbb{E}(X|B)\chi_B + \mathbb{E}(X|(\Omega \setminus B))\chi_{\Omega \backslash B}. \tag{2.6.1}$$

Not that if $\mathcal{B} = \{\emptyset, B, \Omega \setminus B, \Omega\}$, the random variable Y satisfies the condition

$$\int_C Y d\mathbb{P} = \int_C X d\mathbb{P} \ for \ all \ C \in \mathcal{B}. \tag{2.6.2}$$

If $\mathcal{B}$ denotes the σ-field $\{\emptyset, B, \Omega \backslash B, \Omega\}$, the random variable Y will be denoted by $\mathbb{E}[X|\mathcal{B}]$. It will be called a conditional expectation of X given $\mathcal{B}$.
Let us list several important properties of conditional expectations. Note that all the equalities and inequalities below hold almost surely.

1. If $f_1, f_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and a, b are constants, then

$$\mathbb{E}[af_1 + bf_2|\mathcal{B}] = a\mathbb{E}[f_1|\mathcal{B}] + b\mathbb{E}[f_2|\mathcal{B}]. \tag{2.6.3}$$

2. If $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{B}_1$ and $\mathcal{B}_2$ are σ-subalgebras of $\mathcal{F}$ such that $\mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{F}$, then

$$\mathbb{E}[f|\mathcal{B}_2] = \mathbb{E}[\mathbb{E}[f|\mathcal{B}_1]|\mathcal{B}_2].$$

3. If $f_1, f_2 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $f_1 \leqslant f_2$, then $\mathbb{E}[f_1|\mathcal{B}] \leqslant \mathbb{E}[f_2|\mathcal{B}]$.

4. $\mathbb{E}[\mathbb{E}[f|\mathcal{B}]] = \mathbb{E}f$.
5. If $g, fg \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and f is measurable with respect to $\mathcal{B}$, then

$$\mathbb{E}[fg|\mathcal{B}] = f\mathbb{E}[g|\mathcal{B}]. \tag{2.6.4}$$

Properties 1-3 are clear. To prove property 4, it suffices to take $A = \Omega$ in the equality $\int_A f d\mathbb{P} = \int_A \mathbb{E}[f|\mathcal{B}]d\mathbb{P}$.

To prove the last property, first we consider the case when f is the indicator function of a set $B \in \mathcal{B}$. Then for any $A \in \mathcal{B}$

$$\int_A \chi_B \mathbb{E}[g|\mathcal{B}]d\mathbb{P} = \int_{A\cap B} \mathbb{E}[g|\mathcal{B}]d\mathbb{P} = \int_{A\cap B} g d\mathbb{P} = \int_A \chi_B g d\mathbb{P},$$

which proves the statement for $f = \chi_B$. By linearity, the statement is also true for simple functions taking a finite number of values. Next, without loss of generality, we may assume that $f, g \geqslant 0$. Then we can find a non-decreasing sequence of simple functions f_n, each taking a finite number of values such that $\lim_{n\to\infty} f_n = f$ almost surely. We have $f_n g \to fg$ almost surely, and the Dominated Convergence Theorem for conditional expectations can be applied to the sequence $f_n g$ to conclude that

$$\mathbb{E}[fg|\mathcal{B}] = \lim_{n\to\infty} \mathbb{E}[f_n g|\mathcal{B}] = \lim_{n\to\infty} f_n \mathbb{E}[g|\mathcal{B}] = f\mathbb{E}[g|\mathcal{B}].$$

We now state Jensen's Inequality and the Conditional Jensen's Inequality, essential to our discussion of conditional expectations and martingales.

Theorem 2.6.1. ***[KS07, the Conditional Jensen's Inequality]***. *Let g be a convex function on $\mathbb{R}^d$ and f a random variable with values in $\mathbb{R}^d$ such that*

$$\mathbb{E}|f|, \mathbb{E}|g(f)| < \infty.$$

Let $\mathcal{B}$ be a σ-subalgebra of $\mathfrak{F}$. Then almost surely

$$g(\mathbb{E}[f|\mathcal{B}]) \leqslant \mathbb{E}[g(f)|\mathcal{B}].$$

We extend [Tay97, Corollary 5.2.5.] from $L^p(\Omega, \mathfrak{F}, \mathbb{P})$, $1 \leqslant p \leqslant +\infty$, to the Orlicz function space $L^\Phi(\Omega, \mathfrak{F}, \mathbb{P})$, where $\Phi \in \Delta_2(\infty)$ as in following:

Corollary 2.6.2. *Let $L^\Phi(\Omega, \mathfrak{F}, \mathbb{P})$ be an Orlicz function space where the Orlicz function $\Phi \in \Delta_2(\infty)$ and $X \in L^\Phi(\Omega, \mathfrak{F}, \mathbb{P})$ be a random variable. Then $\mathbb{E}[X|\mathcal{B}] \in L^\Phi(\Omega, \mathcal{B}, \mathbb{P})$ where $\mathcal{B}$ is a σ-subalgebra of $\mathfrak{F}$ and*

$$\|\mathbb{E}[X|\mathcal{B}]\|_\Phi \leqslant \|X\|_\Phi$$

Proof. Let Φ be an Orlicz function, then it is convex. By using the Conditional Jensen's Inequality, we have

$$\Phi|\mathbb{E}[X|\mathcal{B}]| \leqslant \Phi\mathbb{E}[|X||\mathcal{B}] \leqslant \mathbb{E}[\Phi|X||\mathcal{B}].$$

Hence,

$$\mathbb{E}[\Phi|\mathbb{E}[X|\mathcal{B}]|] \leqslant \mathbb{E}[\mathbb{E}[\Phi|X||\mathcal{B}]] = \mathbb{E}[\Phi|X|] \tag{2.6.5}$$

Therefore,

$$\|\mathbb{E}[X|\mathcal{B}]\|_\Phi \leqslant \|X\|_\Phi < \infty$$

□

Let $(\Omega, \mathfrak{F})$ be a measurable space and T a subset of $\mathbb{R}$ or $\mathbb{Z}$. We recall basic definitions about the martingale sequences.

Definition 2.6.3. [KS07, Definition 13.8.]. A collection of sub σ-algebras $\mathfrak{F}_t \subseteq \mathfrak{F}, t \in T$, is called a filtration if $\mathfrak{F}_s \subseteq \mathfrak{F}_t$ for all $s \leqslant t$.

Definition 2.6.4. [KS07, Definition 13.14.]. A random process X_t is called adapted to a filtration $\mathfrak{F}_t$ if X_t is $\mathfrak{F}_t$-measurable for each $t \subseteq T$.

Now, we are ready to mention the definition of the martingale sequence.

Definition 2.6.5. [KS07, Definition 13.16.]. A family $(X_t, \mathfrak{F}_t)$, $t \in T$ is called a *martingale* if the process X_t is adapted to the filtration $\mathfrak{F}_t$, $X_t \in L^1(\Omega, \mathfrak{F}, P)$ for all t, and

$$X_s = \mathbb{E}[X_t|\mathfrak{F}_s] \ for \ s \leqslant t.$$

If the equal sign is replaced by $\leqslant$ or $\geqslant$, then $(X_t, \mathfrak{F}_t)$ $t \in T$ is called a submartingale or supermartingale respectively.

We shall often say that X_t is a martingale, without specifying a filtration, if it is clear from the context what the parameter set and the filtration are. If $(X_t, \mathfrak{F}_t)_{t\in T}$ is a martingale and f is a convex function such that $f(X_t)$ is integrable for all t, then $(f(X_t), \mathfrak{F}_t)_{t\in T}$ is a submartingale. Indeed, by the Conditional Jensen's Inequality,

$$f(X_s) = f(\mathbb{E}[X_t|\mathfrak{F}_s]) \leqslant \mathbb{E}[f(X_t)|\mathfrak{F}_s].$$

For example, if $(X_t, \mathfrak{F}_t)_{t\in T}$ is a martingale, then $(|X_t|, \mathfrak{F}_t)_{t\in T}$ is a submartingale. The right-closable martingale sequence is a special case of the martingale sequences that will be used throughout the following part of this chapter

Definition 2.6.6. [KS07, Definition 13.31.]. A martingale $(Xn, \mathfrak{F}_n)_{n\in\mathbb{N}}$ is said to be right-closable if there is a random variable $X_\infty \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\mathbb{E}[X_\infty|\mathfrak{F}_n] = X_n$ for all $n \in \mathbb{N}$. The random variable $X\infty$ is sometimes referred to as the last element of the martingale. We can define $\mathfrak{F}_\infty$ as the minimal σ-algebra containing $\mathfrak{F}_n$ for all n. For a right-closable martingale we can define $X'_\infty = \mathbb{E}[X_\infty|\mathfrak{F}_\infty]$. Then X'_∞ also serves as the last element since

$$\mathbb{E}[X'_\infty|\mathfrak{F}_n] = \mathbb{E}[\mathbb{E}[X_\infty|\mathfrak{F}_\infty]|\mathfrak{F}_n] = \mathbb{E}[X_\infty|\mathfrak{F}_n] = X_n.$$

Therefore, without loss of generality, we shall assume from now on that, for a right-closable martingale, the last element X_∞ is $\mathfrak{F}_\infty$-measurable.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a fixed probability space and assume $(\mathfrak{F}_n)_n$ is an increasing sequence of sub σ-algebra of $\mathfrak{F}$ and $\mathfrak{F} = \bigvee_{n=0}^{\infty} \mathfrak{F}_n$ (i.e, $\mathfrak{F}$ is the minimal sub σ-algebra containing $\mathfrak{F}_n$ for all $n \geqslant 0$). If $f \in L^1(\Omega)$, we let $f_n = \mathbb{E}[f|\mathfrak{F}_n]$ and assume $f_0 = 0$. The sequence $(f_n)_n$ is a closable martingale and the corresponding martingale difference sequence is given by $\Delta f_n = f_n - f_{n-1}$.
Define for each n

$$f_n^* = \max_{1\leqslant k\leqslant n} |f_k|, \ \ S_n(f) = (\sum_{k=1}^{n} [\Delta f_k]^2)^{\frac{1}{2}}.$$

The maximal function f^* and the square function $S(f)$ are given by

$$f^* = \lim_{n\to\infty} f_n^*, \ \ S(f) = \lim_{n\to\infty} S_n(f).$$

It is well known that if Φ is an Orlicz function with $\inf\limits_{t\geqslant 0} \frac{t\Phi'(t)}{\Phi(t)} > 1$ (i.e., $\Phi \in \nabla_2$ by [RR02, Theorem (I.2), page 3]), the extension of the classical Doob's inequality is given by the inequality:

$$\sup_{n\geqslant 0}\|f_n\|_\Phi \leqslant \|f^*\|_\Phi \leqslant c \sup_{n\geqslant 0}\|f_n\|_\Phi \tag{2.6.6}$$

where c is a constant depending only on Φ, (see [TL02] and [AR06]). Since $\|f\|_\Phi = \sup\limits_{n\geqslant 0}\|f_n\|_\Phi$, then we have:

Proposition 2.6.7. *Let $f \in L^\Phi(\Omega, \mathfrak{F}, \mathbb{P})$ where Φ is an Orlicz function satisfying $\Delta_2(\infty) \cap \nabla_2$, and $(\mathfrak{F}_n)_n$ be an increasing sequence of sub σ-algebra of $\mathfrak{F}$ and $\mathfrak{F} = \bigvee_{n=0}^{\infty} \mathfrak{F}_n$. Then*

$$\|f\|_\Phi \leqslant \|f^*\|_\Phi \leqslant c\ \|f\|_\Phi \tag{2.6.7}$$

Theorem 2.6.8. ***[BDG72, Theorem 1.1]**. Suppose Φ is a convex function from $[0,\infty)$ into $[0,\infty)$ such that $\Phi(0) = 0$ with the growth condition $\Phi(2\lambda) \leqslant b_0\Phi(\lambda)$, $(\lambda > 0)$ set $\Phi(\infty) = \lim\limits_{\lambda\to\infty} \Phi(\lambda)$. Then*

$$B_1\mathbb{E}\Phi(S(f)) \leqslant \mathbb{E}\Phi(f^*) \leqslant B_2\mathbb{E}\Phi(S(f)) \tag{2.6.8}$$

for all martingales f. The constants B_1 and B_2 depend only on b_0.

From this Theorem one can conclude that there is $K_1 \geqslant 1$ s.t

$$K_1^{-1}\|S(f)\|_\Phi \leqslant \|f^*\|_\Phi \leqslant K_1\|S(f)\|_\Phi \tag{2.6.9}$$

Proposition (2.6.7) and Theorem (2.6.8) imply to Burkholder-Gundy inequality:

Proposition 2.6.9. *Let Φ be an Orlicz function such that $\Phi \in \Delta_2 \cap \nabla_2$ then there exists a constant $K_2 \geqslant 1$ such that for all $f \in L^\Phi(\Omega, \mathfrak{F}, \mathbb{P})$*

$$K_2^{-1}\|f\|_\Phi \leqslant \|S(f)\|_\Phi \leqslant K_2\|f\|_\Phi. \tag{2.6.10}$$

Proposition 2.6.10. ***[JMST79, Proposition 9.5]**. Let X be a rearrangement invariant function space on $[0,1]$ such that the Boyd's indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Suppose $\beta_1 \subset \beta_2 \subset \cdots$ are sub-σ-algebras and $f_1, f_2, \cdots$ are measurable functions, then there is a constant K_3 s.t*

$$\|(\sum \mathbb{E}[f_j|\beta_j]^2)^{\frac{1}{2}}\|_X \leqslant K_3\|(\sum f_j^2)^{\frac{1}{2}}\|_X \tag{2.6.11}$$

2.7 Complemented embedding of separable rearrangement invariant function spaces into spaces with unconditional Schauder decompositions

We aim to extend [BRS81, Theorem 1.1] to the r.i. function space $X[0,1]$ with the Boyd indices $0 < \beta_X \leq \alpha_X < 1$. Our proof heavily relies on the proof of [BRS81]. We will use interpolation arguments to extend it.

Theorem 2.7.1. *Let $X[0,1]$ be a r.i. function space whose Boyd indices satisfy $0 < \beta_X \leq \alpha_X < 1$. Suppose $X[0,1]$ is isomorphic to a complemented subspace of a Banach space Y with an unconditional Schauder decomposition (Y_j). Then one of the following holds:*

(1) There is an i so that $X[0,1]$ is isomorphic to a complemented subspace of Y_i.

(2) A block basic sequence of the Y_i's is equivalent to the Haar basis of $X[0,1]$ and has closed linear span complemented in Y.

We first need some facts about unconditional bases and decompositions that were mentioned in [BRS81]. Given a Banach space B with unconditional basis (b_i) and (x_i) a sequence of non-zero elements in B, say that (x_i) is disjoint if there exist disjoint subsets $M_1, M_2, \cdots$ of $\mathbb{N}$ with $x_i \in [b_j]_{j\in M_i}$ for all i. Say that (x_i) is essentially disjoint if there exists a disjoint sequence (y_i) such that $\Sigma\|x_i - y_i\|/\|x_i\| < \infty$. If (x_i) is essentially disjoint, then (x_i) is essentially a block basis of a permutation of (b_i). Also, (x_i) is unconditional basic sequence. Throughout this paper, if $\{b_i\}_{i\in I}$ is an indexed family of elements of a Banach space B, $[b_i]_{i\in I}$ denotes the closed linear span of $\{b_i\}_{i\in I}$ in B. We recall the definition of the Haar system (h_n) which is normalized in L^∞: let $h_1 \equiv 1$ and for $n = 2^k + j$ with $k \geqslant 0$ and $1 \leqslant j \leqslant 2^k$,

$$h_n = \chi_{[\frac{j-1}{2^k}, \frac{2j-1}{2^{k+1}})} - \chi_{[\frac{2j-1}{2^{k+1}}, \frac{j}{2^k})}.$$

Moreover, the Haar system is unconditional basis of the r.i. function space $X[0,1]$ if and only if the Boyd indices of $X[0,1]$ satisfy $0 < \beta_X \leqslant \alpha_X < 1$, see [LT79, Theorem 2.c.6]. We use $[f = a]$ for $\{t;\ f(t) = a\}$; μ is the Lebesgue measure. For a measurable function f, *supp* $f\ = [f \neq 0]$.
The following results are recalled in [BRS81], and as noted in [BRS81] it is a rephrasing of Lemma 1.1 of [AEO77].

Lemma 2.7.2. ***[BRS81, Lemma 1.2.].*** *Let (b_n) be an unconditional basis for the Banach space B with biorthogonal functionals (b_n^*), $T : B \mapsto B$ an operator, $\epsilon > 0$, and (b_{n_i}) a subsequence of (b_n) so that (Tb_{n_i}) is essentially disjoint and $|b_{n_i}^* Tb_{n_i}| \geqslant \epsilon$ for all i. Then (Tb_{n_i}) is equivalent to (b_{n_i}) and $[Tb_{n_i}]$ is complemented in B.*

If (X_i) is an unconditional Schauder decomposition, say that P_i is the natural projection onto X_i if $P_i x = x_i$ provided $x = \sum x_j$ with $x_j \in X_j$. We refer to (P_i) as the projections corresponding to the decomposition.

Lemma 2.7.3. ***[BRS81, Lemma 1.3.].*** *Let Z and Y be Banach spaces with unconditional Schauder decompositions (Z_i) and (Y_i) respectively; and let (P_i) (resp. Q_i) be the natural projection from Z (resp. Y) onto Z_i (resp. Y_i). Then if $T : Z \mapsto Y$ is a bounded linear operator, so is $\sum Q_i T P_i$.*

The next result is the Scholium 1.4 of the [BRS81]. It is used directly in the proof of case 2 of the main theorem in this section. Throughout this section, "projection" means "bounded linear projection" and "operator" means "bounded linear operator".

Scholium 2.7.4. *Let Y be Banach space with unconditional Schauder decomposition with corresponding projections (Q_i), and let Z be a complemented subspace of Y with unconditional basis (z_i) with biorthogonal functionals (z_i^*). Suppose there exist $\epsilon > 0$, a projection $U : Y \mapsto Z$ and disjoint subsets $M_1, M_2, \cdots$ of $\mathbb{N}$ with the following properties:*
(a) $(UQ_i z_l)_{l\in M_i, i\in\mathbb{N}}$ is essentially disjoint sequence.
(b) $|z_l^*(UQ_i z_l)| \geqslant \epsilon$ for all $l \in M_i$, $i \in \mathbb{N}$.
Then $(Q_i z_l)_{l\in M_i, i\in\mathbb{N}}$ is equivalent to $(z_l)_{l\in M_i, i\in\mathbb{N}}$ and $[Q_i z_l]_{l\in M_i, i\in\mathbb{N}}$ is complemented in Y.

Proof. Let $M = \cup_{i=1}^{\infty} M_i$ and $L = \mathbb{N} \setminus M$. Let $X_i = [x_l]_{l\in M_i}$ for $i > 1$ and $X_1 = [x_l]_{l\in M_1\cup L}$. Then (X_i) is an unconditional Schauder decomposition for X; let (P_i) be the corresponding projections. Also, let T be the natural projection from X onto $[x_l]_{l\in M}$. Now, if we regard

T as an operator from X into Y, $V = \sum Q_i T P_i$ is also an operator, by Lemma 2.7.3. Fixing i and $l \in M_i$, we have

$$V(x_l) = Q_i T P_i(x_l) = Q_i(x_l). \tag{2.7.1}$$

Hence by (b),

$$|x_l^* UV(x_l)| = |x_l^* UV(x_l)| \geq \epsilon. \tag{2.7.2}$$

Moreover, $(UV(x_l))_{l\in M}$ is almost disjoint by (a).
Thus, Lemma 2.7.2 applies and $(UV(x_l))_{l\in M}$ is equivalent to $(x_l)_{l\in M}$ and $[UV(x_l)]_{l\in M}$ is complemented in X. It now follows directly that $(V(x_l))_{l\in M}$ is equivalent to $(x_l)_{l\in M}$ with $[V(x_l)]_{l\in M}$ complemented in Y. To see the final assertion, $V(x_l)$ is dominated by (x_l) but dominates $(UV(x_l))$, hence $V(x_l)$ is equivalent to (x_l). Let P be a projection from X onto $[UV(x_l)]_{l\in M}$ and let $S : [UV(x_l)]_{l\in M} \longrightarrow [x_l]_{l\in M}$ be the isomorphism with $SUV(x_l) = x_l$ for all $l \in M$. Then $Q = VSPU$ is a projection from Y onto $[V(x_l)]_{l\in M}$. □

The next Lemma is proved in [LT79, Theorem (2.d.10)] and it is the extension of the fundamental result of Gamlen and Gaudet [GG73] to separable r.i function spaces $X[0,1]$.

Lemma 2.7.5. *Let $I \subset \mathbb{N}$ such that if*

$$E = \{t \in [0,1]; t \in supp\ h_i\ for\ infinitely\ many\ i \in I\},$$

then E is of positive Lebesgue measure. Then $[h_i]_{i\in I}$ is isomorphic to $X[0,1]$.

Recall that $X(\ell^2)$ is the completion of the space of all sequences $(f_1, f_2, \cdots)$ of elements of X which are eventually zero, with respect to the norm

$$\|(f_1, f_2, \cdots)\|_{X(\ell^2)} = \|(\sum |f_i|^2)^{\frac{1}{2}}\|_X.$$

In what follows, we let $X[0,1]$ be a separable r.i. function space with $0 < \beta_X \leqslant \alpha_X < 1$.

Let $\{h_i\}_i$ be the Haar basis of $X[0,1]$, fix i and let (h_{ij}) be the element of $X(\ell^2)$ whose j-th coordinate equals h_i, all other coordinates 0. Then $(h_{ij})_{i,j}$ is an unconditional basis for $X(\ell^2)$, [LT79, Proposition 2.d.8]. We recall [JMST79, Lemma 9.7] that states: Let X be a r.i. function space on $[0,1]$ with $0 < \beta_X \leq \alpha_X < 1$. Then there exists a constant K such that

$$K^{-1}\|\sum_{i=1}^{\infty} a_i h_i\|_X \leq \|\left(\sum_{i=1}^{\infty}(a_i h_i)^2\right)^{1/2}\|_X \leq K\|\sum_{i=1}^{\infty} a_i h_i\|_X. \tag{2.7.3}$$

for every sequence $\{a_i\}_{i\geq 1}$ of scalars.
The following result is a direct consequence of [JMST79, Lemma 9.7].

Scholium 2.7.6. *There is a constant K so that for any function $j : \mathbb{N} \mapsto \mathbb{N}$, $(h_{ij(i)})_{i\geqslant 1}$ in $X(l_2)$ is $K-$ equivalent to (h_i) in X.*

The following is a consequence of the proof of [LT79, Theorem (2.d.11)] that $X[0,1]$ is primary. Let $(h^*_{ij})_{i,j}$ denote the biorthogonal functionals to $(h_{ij})_{i,j}$.

Scholium 2.7.7. *Let $T : X(l_2) \mapsto X(l_2)$ be a given operator. Suppose there is a $c > 0$ so that when $I = \{i : |h^*_{ij} T h_{ij}| \geqslant c$ for infinitely many $j\}$, then E has positive Lebesgue measure, where*

$$E = \{t \in [0,1];\ t \in supp\, h_i \text{ for infinitely many } \ i \in I\}$$

Then there is a subspace Y of $X(l_2)$ with Y isomorphic to X, $T|Y$ an isomorphism, and TY complemented in $X(l_2)$.

Proof. Fix $i \in I$, by the definition of I, there is a sequence $j_1 < j_2 < \cdots$ with $\|Th_{ij_k}\| \geqslant c > 0$ for all k. Since $\{h_{ij_k}\}_{k=1}^{\infty}$ is equivalent to the unit vectors in l_2 then it is weakly null and so $\{Th_{ij_k}\}_{k\geqslant 1}$ is weakly null. Thus, there exists $j : I \mapsto \mathbb{N}$ such that $\{Th_{ij(i)}\}_{i\in I}$ is equivalent to a block basis $(z_i)_{i\geqslant 1}$ and we can choose it such that $\sum_{i\in I} \frac{\|z_i - Th_{ij(i)}\|}{\|Th_{ij(i)}\|} < \infty$ by [BP58, Theorem (3)]. Thus $\{Th_{ij(i)}\}_{i\in I}$ is essentially disjoint with respect to $\{h_{ij}\}_{i,j=1}^{\infty}$ and $|h^*_{ij(i)}Th_{ij(i)}| \geqslant c$. Then by Lemma (2.7.2), $[Th_{ij(i)}]_{i\in I}$ is complemented in $X(l_2)$, and $(Th_{ij(i)})_{i\in I}$ is equivalent to $(h_{ij(i)})_{i\in I}$, which is equivalent to $(h_i)_{i\in I}$ by Scholium (2.7.6). In turn, $[h_i]_{i\in I}$ is isomorphic to $X[0,1]$, by Lemma (2.7.5). □

Corollary 2.7.8. *Let $T : X[0,1] \mapsto X[0,1]$ be a given operator. Then for $S = T$ or $I - T$, there exists a subspace Y of $X[0,1]$ with Y isomorphic to $X[0,1]$, $S|Y$ an isomorphism, and $S(Y)$ complemented in $X[0,1]$.*

Proof. Since $X[0,1]$ is isomorphic to $X(l_2)$ by [LT79, Proposition 2.d.4], we can prove the statement with respect to $X(l_2)$. For each i, j, at least one of the numbers $|h^*_{ij}Th_{ij}|$ and $|h^*_{ij}(I-T)h_{ij}|$ is $\geqslant \frac{1}{2}$. Let $I_1 = \{i : |h^*_{ij}Th_{ij}| \geqslant \frac{1}{2}$ *for infinitely many* $j\}$, $I_2 = \{i : |h^*_{ij}(I-T)h_{ij}| \geqslant \frac{1}{2}$ *for infinitely many* $j\}$. Then $\mathbb{N} = I_1 \bigcup I_2$, hence, for $k = 1$ *or* 2, $E_k = \{t \in [0,1];\ t \in$ *supp* h_i *for infinitely many* $i \in I_k\}$ has positive Lebesgue measure and by Scholium (2.7.7) we get the result. □

Theorem 2.7.9. *Let Z and Y be given Banach spaces. If $X[0,1]$ is isomorphic to a complemented subspace of $Z \oplus Y$, then $X[0,1]$ is isomorphic to a complemented subspace of Z or to a complemented subspace of Y.*

Proof. Let P (resp. Q) denote the natural projection from $Z \bigoplus Y$ onto Z (resp. Y). Hence $P + Q = I$. Let K be a complemented subspace of $Z \bigoplus Y$ isomorphic to $X[0,1]$ and let $U : Z \bigoplus Y \mapsto K$ be a projection. Since $UP|K + UQ|K = I|K$, Corollary (2.7.8) shows that there is a subspace W of K with W isomorphic to $X[0,1]$, $T|W$ an isomorphism, and TW complemented in K, where $T = UP|K$ or $T = UQ|K$. Suppose for instance the former: Let S be a projection from K onto TW and $R = (T|W)^{-1}$. Then $I|W = RSUP|W$; hence since the identity on W may be factored through Z, W is isomorphic to a complemented subspace of Z. □

The next lemma is [GG73, Lemma 4] and it is pointed out in the proof of [LT79, Theorem (2.d.10)] for separable r.i function space.

Lemma 2.7.10. *Let (x_i) be a measurable functions on $[0,1]$ with x_1 $\{0,1\}-$valued and x_i $\{1,0,-1\}-$ valued for $i > 1$. Suppose there exist positive constants a and b so that, for all positive l, with k the unique integer, $1 \leqslant k \leqslant l$, and α the unique choice of $+1$ or -1 so that supp $h_{l+1} = [h_k = \alpha]$, then*

(a) $[x_k = \alpha] =$ supp x_{l+1}

(b) $\frac{a}{2}\int |h_k| \leqslant \mu([x_{l+1} = \beta]) \leqslant \frac{b}{2}\int |h_k|$ for $\beta = \pm 1$

Then (x_n) is equivalent to (h_n) in $X[0,1]$, and $[x_n]$ is the range of a one-norm projection defined on $X[0,1]$.

Proof. Let $A_n =$ *supp* x_n, then the subspace $[x_n]$ is complemented in $X[0,1]$ since it is the range of the projection $P(f) = \chi_A \mathbb{E}_{\mathcal{F}}(f)$ of norm one, where $A = \bigcup_{n=1}^{\infty} A_n$ and $\mathbb{E}_{\mathcal{F}}$ denotes the conditional expectation operator with respect to the sub-σ-algebra generated

by the measurable sets A_n.
Since (x_n) and (h_n) are equivalent in $L^p[0,1]$ for all $1 < p < \infty$, by [GG73, Lemma 4], then the operator $R_1 : L^p[0,1] \mapsto L^p[0,1]$ defined by $R_1(\sum_{n=1}^{\infty} a_n h_n) = \sum_{n=1}^{\infty} a_n x_n$ is bounded for all $1 < p < \infty$. Therefore, the Boyd interpolation theorem implies that R_1 will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leqslant \alpha_X < 1$. Now, we will define a bounded operator R_2 on $L^p[0,1]$ as follows: if $P(f) = \sum_{n=1}^{\infty} a_n x_n$, then $R_2(f) = \sum_{n=1}^{\infty} a_n h_n$. Again, the Boyd interpolation theorem implies that R_2 will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leqslant \alpha_X < 1$, and this clearly yields the equivalence of the sequences (x_n) and (h_n) in $X[0,1]$. □

Scholium 2.7.11. *Let (z_i) be a sequence of measurable functions on $[0,1]$ such that z_1 is $\{0,1\}$-valued nonzero in L^1 and z_i is $\{0,-1,1\}$- valued with $\int z_i = 0$ for all $i > 1$. Suppose that for all positive l, letting k be the unique integer, $1 \leqslant k \leqslant l$, and α the unique choice of 1 or -1 so that supp $h_{l+1} = \{t; h_k(t) = \alpha\}$, then*

$$\mathit{supp}\, z_{l+1} \subset \{t; z_k(t) = \alpha\}$$

and $\mu(\{t; z_k(t) = \alpha\} \setminus \mathit{supp}\ z_{l+1}) \leqslant \epsilon_l \int |z_1|$, where $\epsilon_l = \frac{1}{2^{j^2}}$. Then (z_n) is equivalent to (h_n), and $[z_n]$ is complemented in X.

Proof. The proof is depending on [BRS81, Scholium (1.11)] and The Boyd interpolation theorem with same procedure in the proof of Lemma (2.7.10). □

Now, We will use the same procedure of [BRS81, Theorem 1.1] in order to prove Theorem (2.7.1).
We assume that $X(\ell^2)$ is a complemented subspace of Y; let $U : Y \mapsto X(\ell^2)$ be a projection. Let $(Y_i)_i$ be an unconditional decomposition of Y. Suppose that (1) fails, that is, there is no i with $X[0,1]$ isomorphic to a complemented subspace of Y_i. We shall then construct a blocking of the decomposition (Y_i) with corresponding projections (Q_i), finite disjoint subsets $M_1,\ M_2, \cdots$ of $\mathbb{N}$, and a map $j : \cup_{i=1}^{\infty} M_i \mapsto \mathbb{N}$ so that:
(i) $(Q_k h_{ij(i)})_{i\in M_k,\, k\in\mathbb{N}}$ is equivalent to $(h_i)_{i\in M_k,\, k\in\mathbb{N}}$ with $[Q_k h_{ij(i)}]_{i\in M_k,\, k\in\mathbb{N}}$ complemented in Y.
(ii) (z_k) is equivalent to the Haar basis and $[z_k]$ is complemented in $X[0,1]$, where $z_k = \sum_{i\in M_k} h_i$ for all k.
We simply let $b_k = \sum_{i\in M_k} Q_k h_{ij(i)}$ for all k; then (b_k) is the desired block basic sequence equivalent to the Haar basis with $[b_k]$ complemented.
Let us now proceed to the construction. Let P_i be the natural projection from Y onto Y_i. More generally, for F a subset of $\mathbb{N}$ we let $P_F = \sum_{i\in F} P_i$. Also, we let $R_n = I - \sum_{i=1}^{n} P_i\ (= P_{(n,\infty)})$. We first draw a consequence from our assumption that no Y_i contains a complemented isomorphic copy of $X[0,1]$.

Lemma 2.7.12. *For each n, let*

$$I = \{i \in \mathbb{N}; h_{ij}^* U R_n h_{ij} > \frac{1}{2} \text{ for infinitely many integers } j\}. \tag{2.7.4}$$

Let $E_I = \{t \in [0,1];$ t belongs to the support of h_i for infinitely many integers $i \in I\}$. Then $\mu(E_I) = 1$ (where μ is the Lebesgue measure).

Indeed, let $L = \{i \in \mathbb{N}; h_{ij}^* U P_{[1,n]} h_{ij} \geqslant \frac{1}{2}$ *for infinitely many integers j*$\}$; then $I \cup L = \mathbb{N}$. Hence $E_I \bigcup E_L = [0,1]$. So if $\mu(E_I) < 1, \mu(E_L) > 0$. But then $T = UP_{[1,n]}$ satisfies the hypotheses of Scholium (2.7.7). Hence there is a subspace Z of $X(\ell^2)$, with Z isomorphic to $X[0,1]$ and TZ complemented in $X(\ell^2)$. It follows easily that $P_{[1,n]}|Z$

is an isomorphism with $P_{[1,n]}Z$ complemented; that is, X[0,1] embeds as a complemented subspace of $Y_1 \bigoplus \cdots \bigoplus Y_n$. Hence by Theorem (2.7.9), $X[0,1]$ embeds as a complemented subspace of Y_i for some i, a contradiction.

Lemma 2.7.13. *Let $I \subset \mathbb{N}$, E_I as in Lemma (2.7.12) with $\mu(E_I) = 1$, and $S \subset [0,1]$ with S a finite union of disjoint left-closed dyadic intervals. Then there exists a $J \subset I$ so that $supp\, h_i \bigcap supp\, h_l = \emptyset$, for all $i \neq l$, $i,\ l \in J$, with $S \supset \cup_{j\in J}\, supp\, h_j$ and $S \setminus \cup_{j\in J}\, supp\, h_j$ of measure zero.*

Proof. It suffices to prove the result for S equal to the left-closed dyadic interval. Now any two Haar functions either have disjoint supports, or the support of one is contained in that of the other. Moreover, for all but finitely many $i \in I$, $supp\ h_i \subset S$ or $supp\, h_i \cap S = \emptyset$. Hence S differs from $\bigcup\{supp\, h_j :\ supp\, h_i \subset S,\ j \in I\}$ by a measure-zero set. Now simply let $J = \{j \in I;\ supp\, h_j \subset S,\ and\ there\ is\ no\ l \in I\ with\ supp\, h_j \subset supp\, h_l \subset S\}$. □

We now choose $M_1,\ M_2, \cdots$ disjoint finite subsets of $\mathbb{N}$, a map $j : \cup_{i=1}^{\infty} M_i \mapsto \mathbb{N}$, and $1 = m_0 < m_1 < m_2 < \cdots$ with the following properties:

- **A**. For each k, the h_i's for $i \in M_k$ are disjointly supported. Set $z_k = \sum_{i\in M_k} h_i$. Then z_k satisfies the hypotheses of Scholium (2.7.11)
- **B**. Let $Q_k = P_{[m_{k-1},m_k]}$ for all k. Then $(UQ_k h_{ij(i)})_{i\in M_k, k\in\mathbb{N}}$ is essentially disjoint and $h^*_{ij(i)} UQ_k h_{ij(i)} > \frac{1}{2}$ for all $i \in M_k,\ k \in \mathbb{N}$.

Having accomplished this, we set $b_k = \sum_{i\in M_k} Q_k h_{ij(i)}$ for all k. Then by B, (b_k) is a block basic sequence of the Y_i's. By Scholium (2.7.4),

$$(Q_k h_{ij(i)})_{i\in M_k,\ k\in\mathbb{N}} \approx (h_{ij(i)})_{i\in M_k,\ k\in\mathbb{N}} \approx (h_i)_{i\in M_k,\ k\in\mathbb{N}} \tag{2.7.5}$$

where " $\approx$ " denotes equivalence of basic sequences; the last equivalence follows from Scholium (2.7.6) i.e., the unconditionality of the Haar basis. Hence by the definitions of (b_k) and (z_k), (b_k) is equivalent to (z_k) which is equivalent to (h_k), the Haar basis, by Scholium (2.7.11). Also, since $[z_k]$ is complemented in $X[0,1]$ by Scholium (2.7.11), $[b_k]$ is complemented in $[Q_k h_{ij(i)}]_{i\in M_k,\ k\in\mathbb{N}}$ by (2.7.5).
Again by Scholium (2.7.4), $[Q_k h_{ij(i)}]_{i\in M_k,\ k\in\mathbb{N}}$ is complemented in Y, hence also $[b_k]$ is complemented in Y.
It remains now to choose the M_i's, m_i's and the map j. To insure **B**, we shall also choose a sequence $(f_i)_{i\in M_k,\ k\in\mathbb{N}}$ of disjointly supported elements of $X(\ell^2)$ (disjointly supported with respect to the basis (h_{ij})) so that

$$\sum_{i\in M_k} \frac{\|UQ_k h_{ij(i)} - f_i\|}{\|UQ_k h_{ij(i)}\|} < \frac{1}{2^k},\ for\ all\ k \tag{2.7.6}$$

To start, we let $M_1 = \{1\}$ and $j(1) = 1$. Thus $z_1 = 1$; we also set $f_1 = h_{11}$. Then $h_{11} = Uh_{11} = lim_{n\to\infty} UP_{[1,n]} h_{11}$. So it is obvious that we can choose $m_1 > 1$ such that $\|UP_{[1,m_1]}h_{11} - h_{11}\| < \frac{1}{2}$; hence $h^*_{11} UP_{[1,m_1]} h_{11} > \frac{1}{2}$. Thus, the first step is essentially trivial.
Now suppose $l \geqslant 1$, $M_1, \cdots, M_l$, $m_1 < \cdots < m_l$, $j : \cup_{i=1}^{l} M_i \mapsto \mathbb{N}$ and $(f_i)_{i\in M_k,\ 1<k<l}$ have been chosen.We set $z_i = \sum_{j\in M_i} h_j$ for all i, $1 \leqslant i \leqslant l$.
Let $1 \leqslant k \leqslant l$ be the unique integer and α the unique choice of ± 1 so that $supp\ h_{l+1} = [h_k = \alpha]$. Let $S = [z_k = \alpha]$. Set $n = m_l$ and let I be as in Lemma (2.7.12). Since S is a finite union of disjoint left-closed dyadic intervals, by Lemma (2.7.13) we may choose

a finite set $M_{l+1} \subset I$, disjoint from $\cup_{i=1}^{l} M_i$, so that the h_i's for $i \in M_{l+1}$ are disjointly supported with $supp\ h_i \subset S$ for $i \in M_{l+1}$

$$\mu(S \setminus \cup_{i \in M_{l+1}} supp\ h_i) \leqslant \epsilon_l \tag{2.7.7}$$

(where $\epsilon_j = \frac{1}{2^{j^2}}$ for all j). At this point, we have that $z_{l+1} = \sum_{i \in M_{l+1}} h_i$ satisfies the conditions of Scholium (2.7.11).
By the definition of I, for each $i \in M_{l+1}$ there is an infinite set J_i with

$$h_{ij}^* U R_n h_{ij} > \frac{1}{2},\ for\ all\ j \in J_i.$$

Now $(UR_n h_{ij})_{j=1}^{\infty}$ is a weakly null sequence; hence it follows that we may choose $j : M_{l+1} \mapsto \mathbb{N}$ and disjointly finitely supported elements $(f_i)_{i \in M_{l+1}}$, with supports (relative to the h_{ij}'s) disjoint from those of $\{f_i : i \in \cup_{i=1}^{l} M_i\}$, so that

$$\sum_{i \in M_{l+1}} \frac{\|UR_n h_{ij(i)} - f_i\|}{\|UR_n h_{ij(i)}\|} < \frac{1}{2^{l+1}}$$

At last, since $R_n g = lim_{k\to\infty} P_{[m_l,k)} g$ for any $g \in X(\ell^2)$, we may choose an $m_{l+1} > m_l$ so that setting $Q_{l+1} = P_{[m_l,m_{l+1})}$, (2.7.6) holds for $k = l+1$ and also

$$h_{ij}^* U Q_k h_{ij(i)} > \frac{1}{2},\ for\ all\ i \in M_k$$

This completes the construction of the M_i's, m_i's and map j. Since (2.7.5) holds, **A** and **B** hold. Thus (2) of Theorem (2.7.1) holds; thus the proof is complete.

2.8 Non-isomorphic complemented subspaces of reflexive Orlicz function spaces

In this section we will extend [Bou81, Theorem (4.30)] to the Orlicz function spaces. We will use in particular the Boyd interpolation theorem and Kalton's result [Kal93], see Theorem (2.4.6)

Again we let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. The space $X(G)$ is a r.i. function space defined on the separable measure space consisting of the Cantor group $G = \{-1,1\}^{\mathcal{C}}$ equipped with the Haar measure. The Walsh functions w_F where F is a finite subset of $\mathcal{C}$ generate the $L^p(G)$ spaces for all $1 \le p < \infty$. Then they also generate the r.i. function space $X(G)$.

We consider a r.i. function space $X[0,1]$ such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. The subspace $X_{\mathcal{C}}$ is the closed linear span in the r.i. function space $X(G)$ over all finite branches Γ in $\mathcal{C}$ of the functions which depend only on the Γ-coordinates. Thus $X_{\mathcal{C}}$ is a subspace of $X(G)$ generated by Walsh functions $\{w_\Gamma = \prod_{c\in\Gamma} r_c;\ \Gamma\ is\ a\ finite\ branch\ of\ \mathcal{C}\}$.

Proposition 2.8.1. *Let $X[0,1]$ be a r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Then $X_{\mathcal{C}}$ is a complemented subspace in $X(G)$.*

Proof. The authors in [Bou81] and [BRS81] express the orthogonal projection P on $X_{\mathcal{C}}^p$ which is bounded in L^p-norm for all $1 < p < \infty$, by taking $\beta_\emptyset =$ trivial algebra and $\beta_c = \mathfrak{G}(d \in \mathcal{C}; d \leqslant c)$ for each $c \in \mathcal{C}$. For $c \in \mathcal{C}$ and $|c| = 1$, let $c' = \emptyset$ and for $c \in \mathcal{C}$ such

that $|c| > 1$, let c' be the predecessor of c in $\mathcal{C}$.
The orthogonal projection P is given by

$$P(f) = \mathbb{E}[f|\beta_{\emptyset}] + \sum_{c\in\mathcal{C}}(\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}]) \tag{2.8.1}$$

for every $f \in L^p(G)$, $1 < p < \infty$.
Let the Boyd indices of $X[0,1]$ satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Then the Boyd interpolation theorem implies that the map P is a bounded projection on $X_{\mathcal{C}}$ for all r.i. function spaces $X(G)$. Therefore, $X_{\mathcal{C}}$ is complemented subspace of $X(G)$. □

The following result directly follows from the previous proposition. However, we mention here another proof by using the martingale sequences and following the steps of Bourgain's proof.

Theorem 2.8.2. *Let $L^{\Phi}(G)$ be an Orlicz function space, where Φ is an Orlicz function such that $\Phi \in \triangle_2 \bigcap \nabla_2$. Then $X^{\Phi}_{\mathcal{C}}$ is complemented in $L^{\Phi}(G)$.*

Proof. We will rewrite the original proof of [Bou81, Theorem (4.30)] in Bourgain's book, in order to extend it to the Orlicz function spaces where Φ is an Orlicz function such that $\Phi \in \triangle_2 \bigcap \nabla_2$.
Let $(\mathcal{E}_i)_i$ be a sequence of sub σ-algebras of G. It is compatible if for all i and j either $\mathcal{E}_i \subseteq \mathcal{E}_j$ or $\mathcal{E}_j \subseteq \mathcal{E}_i$ holds. It is clear that Proposition 2.6.10 holds for compatible sequences $(\mathcal{E}_i)_i$ as well. Indeed, fix non-negative measurable functions $f_1, \dots, f_n$. The compatibility of the $\mathcal{E}_i$'s implies that there is a permutation κ of $\{1, \dots, n\}$ with $\mathcal{E}_{\kappa(i)} \subseteq \mathcal{E}_{\kappa(j)}$ for all $1 \leqslant i < j \leqslant n$. Hence

$$\|(\sum_i \mathbb{E}[f_i|\mathcal{E}_i]^2)^{\frac{1}{2}}\|_{\Phi} = \|(\sum_i \mathbb{E}[f_{\kappa(i)}|\mathcal{E}_{\kappa(i)}]^2)^{\frac{1}{2}}\|_{\Phi} \leqslant K_3\|(\sum_i f^2_{\kappa(i)})^{\frac{1}{2}}\|_{\Phi} = K_3\|(\sum_i f_i^2)^{\frac{1}{2}}\|_{\Phi}.$$

We recall definitions of the sub-σ-algebras in Bourgain's proof.
Let $C_0 = \bigcup_{n=1}^{N}\{1, 2, 3, \cdots, N\}^n$, for some positive integer N. For $c = (c_1, c_2, \cdots, c_n) \in C_0$, put

$$\gamma(c) = \tfrac{N^n - N}{N-1} + \textstyle\sum_{i=1}^{n}(c_i - 1)N^{n-i} + 1.$$

This is an explicit order-preserving enumeration γ of C_0.
Next, we introduce three systems of sub-σ-algebras.
for $i = 0, 1, \cdots, \frac{N^{N+1}-N}{N-1}$, take

$$\mathfrak{F}_{\mathrm{i}} = \mathfrak{G}(c \in C_0; \gamma(c) \leqslant i).$$

for all maximal complexes c in C_0 (i.e., $c \in C_0,\ |c| = N$), define

$$\mathcal{E}'_c = \mathfrak{G}(F'_c)$$

where

$$F'_c = \cup_{n=1}^{N}\{d \in C_0;\ |d| = n,\ \gamma(d) \leqslant \gamma(c\,|\,n)\}.$$

Also, we define

$$\mathcal{E}''_c = \mathfrak{G}(F''_c)$$

where

$$F''_c = \cup_{n=1}^{N}\{d \in C_0;\ |d| = n,\ \gamma(d) \geqslant \gamma(c\,|\,n)\}.$$

It is easily verified that $(\mathfrak{F}_i)_i$ is increasing and both families $(\mathcal{E}'_c)_{c\ maximal}$ and $(\mathcal{E}''_c)_{c\ maximal}$ are compatible. We will now express the orthogonal projection P_0 on $X^{\Phi}_{C_0}$. First we need to define the following sub σ-algebras Take $\beta_\emptyset =$ trivial algebra and $\beta_c = \mathfrak{G}(d \in C_0; d \leqslant c)$ for each $c \in C_0$. For $c \in C_0$ and $|c| = 1$, let $c' = \emptyset$. For $c \in C_0$ and $|c| > 1$, let c' be the predecessor of c in C_0. The orthogonal projection $P_0 : L^{\Phi}(G) \mapsto X^{\Phi}_{C_0}$ is given for every $f \in L^{\Phi}(G)$ by

$$P_0(f) = \mathbb{E}[f|\beta_\emptyset] + \sum_{c\in C_0} (\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}]). \tag{2.8.2}$$

Take $c \in C_0$ and let $i = \gamma(c)$. Clearly, $\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}]$ is $\mathfrak{F}_i$-measurable and $\mathbb{E}[(\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}])|\mathfrak{F}_{i-1}] = 0$. Indeed, since

$$\beta_c \bigcap \mathfrak{F}_{i-1} = \beta_{c'} \bigcap \mathfrak{F}_{i-1} = \beta_{c'},$$

we have

$$\mathbb{E}[\mathbb{E}[f|\beta_c]|\mathfrak{F}_{i-1}] = \mathbb{E}[\mathbb{E}[f|\beta_{c'}]|\mathfrak{F}_{i-1}] = \mathbb{E}[f|\beta_{c'}].$$

So in (2.8.2) we get that $P_0(f)$ is written as a sum of a martingales difference sequence. Application of the Burkholder-Gundy inequality (2.6.10) yields the estimate

$$\|P_0 f\|_\Phi \leqslant K_2 \left\| \left(\mathbb{E}[f|\beta_\emptyset]^2 + \sum_{c\in C_0} (\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}])^2 \right)^{\frac{1}{2}} \right\|_\Phi \tag{2.8.3}$$

so it remains to estimate the right side of (2.8.3).
By the reverse Burkholder-Gundy inequality, we get:

$$\| \left(g_0^2 + \sum g_i^2 \right)^{\frac{1}{2}} \|_\Phi \leqslant K_2 \|f\|_\Phi \tag{2.8.4}$$

taking $g_0 = \mathbb{E}[f|\beta_\emptyset]$ and $g_i = \mathbb{E}[f|\mathfrak{F}_i] - \mathbb{E}[f|\mathfrak{F}_{i-1}]$ for $i \geqslant 1$.
to each $i = 1, \cdots, \frac{N^{N+1}-N}{N-1}$, we associate a maximal complex $\iota(i)$ in C_0 which succeeds to $\gamma^{-1}(i)$, thus such that $\gamma^{-1}(i) \leqslant \iota(i)$, both sequences $(\mathcal{E}'_{\iota(i)})$ and $(\mathcal{E}''_{\iota(i)})$ are compatible. Furthermore,

$$\mathbb{E}[\mathbb{E}[g_i|\mathcal{E}'_{\iota(i)}]|\mathcal{E}''_{\iota(i)}] = \mathbb{E}[g_i|\mathcal{E}'_{\iota(i)} \bigcap \mathcal{E}''_{\iota(i)}].$$

By Proposition (2.6.10) and inequality (2.8.4):

$$\left\| \left(g_0^2 + \sum_{i\geqslant 1} \left(\mathbb{E}[g_i|\mathcal{E}'_{\iota(i)} \bigcap \mathcal{E}''_{\iota(i)}] \right)^2 \right)^{\frac{1}{2}} \right\|_\Phi \leqslant K_3 \left\| \left(g_0^2 + \sum g_i^2 \right)^{\frac{1}{2}} \right\|_\Phi \leqslant K_3 K_2 \|f\|_\Phi. \tag{2.8.5}$$

Now for each i holds:

$$\mathcal{E}'_{\iota(i)} \bigcap \mathcal{E}''_{\iota(i)} = \mathfrak{G}(F'_{\iota(i)} \bigcap F''_{\iota(i)}) = \mathfrak{G}(d \in C_0;\ d \leqslant \iota(i)) = \beta_{\iota(i)}.$$

and taking $c \in C_0$ with $\gamma(c) = i$:

$$\beta_{\iota(i)} \cap \mathfrak{F}_i = \mathfrak{G}(d \in C_0;\ d \leqslant \iota(i)\ and\ \gamma(d) \leqslant i) = \mathfrak{G}(d \in C_0;\ d \leqslant c) = \beta_c.$$

$$\beta_{\iota(i)} \cap \mathfrak{F}_{i-1} = \mathfrak{G}(d \in C_0;\ d \leqslant \iota(i)\ and\ \gamma(d) < i) = \mathfrak{G}(d \in C_0;\ d \leqslant c') = \beta_{c'}.$$

Hence,

$$\mathbb{E}[g_i|\mathcal{E}'_{\iota(i)} \bigcap \mathcal{E}''_{\iota(i)}] = \mathbb{E}[g_i|\beta_{\iota(i)}] = \mathbb{E}[f|\mathfrak{F}_i \bigcap \beta_{\iota(i)}] - \mathbb{E}[f|\mathfrak{F}_{i-1} \bigcap \beta_{\iota(i)}] = \mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}].$$

Therefore, inequality (2.8.5) leads to

$$\left\| \left(\mathbb{E}[f|\beta_\emptyset]^2 + \sum_{c\in C_0} (\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}])^2 \right)^{\frac{1}{2}} \right\|_\Phi \leqslant K_2K_3\|f\|_\Phi. \tag{2.8.6}$$

Consequently, by inequality (2.8.3), we get:

$$\|P_0 f\|_\Phi \leqslant K_3K_2^2\|f\|_\Phi \tag{2.8.7}$$

If we let $N \to \infty$, then the projection P on $X_{\mathcal{C}}^\Phi$, defined by

$$P(f) = \mathbb{E}[f|\beta_\emptyset] + \sum_{c\in\mathcal{C}} (\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}]). \tag{2.8.8}$$

is bounded. □

Since the elements of any finite subset of an infinite branch Γ_∞ are mutually comparable, this subset is a branch. Thus, the subspace X_{Γ_∞} is isometrically isomorphic to $X(\{-1,1\}^{\Gamma_\infty})$ and is a one-complemented subspace of $X(G)$ by the conditional expectation operator.

For a tree T of $\mathcal{C}$, we define the subspace X_T of $X(G)$ as a closed linear span in $X(G)$ over all finite branches Γ in T of all those functions in $X(G)$ which depend only on the coordinates of Γ.
By using the conditional expectation with respect to the sub-σ-algebra generated by a tree T of $\mathcal{C}$, one can find that X_T is a one-complemented subspace of $X_{\mathcal{C}}$ and so it is a complemented subspace of $X(G)$. Therefore, the next result follows from Proposition (2.8.1).

Theorem 2.8.3. *Let $X[0,1]$ be a r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$, and T be a tree on $\mathbb{N}$. Then X_T is a complemented subspace of $X(G)$.*

The next proposition is a direct consequence of Corollary (2.5.5).

Proposition 2.8.4. *Let $X[0,1]$ be a r.i. function space such that $X[0,1]$ is q-concave for some $q < \infty$, the index $\alpha_X < 1$ and the Haar system in $X[0,1]$ is not equivalent to a sequence of disjoint functions in $X[0,1]$. Then $X_{\mathcal{C}}$ is isomorphic to $X(G)$.*

Theorem 2.8.5. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $L^\Phi[0,1]$ does not embed in X_T^Φ if and only if T is a well founded tree.*

Proof. If T contains an infinite branch, then obviously $L^\Phi[0,1]$ embeds in X_T^Φ.
We want to show that if T is well founded, then $L^\Phi[0,1]$ does not embed in X_T^Φ. We will use Theorem (2.5.4) and Theorem (2.7.1): We proceed by induction on $\circ[T]$. Assume the conclusion fails. Let T be a well founded tree such that $\circ[T] = \alpha$ and $L^\Phi[0,1]$ embeds in X_T^Φ, where $\alpha = \min\{\circ[T];\ L^\Phi[0,1] \textit{ embeds in } X_T^\Phi\}$. We write $T = \bigcup_n (n, T_n)$, with $\circ[T] = \sup_n(\circ[T_n] + 1)$.

The space X_T^Φ is generated by the sequence of probabilistically mutually independent spaces $B_n = X_{(n,T_n)}^\Phi$. In particular, $\oplus_n B_n$ is an unconditional decomposition of X_T^Φ by the inequality in [BG70, Corollary(5.4)] that is: let $x_1, x_2, \cdots$ be an independent sequence of random variables, each with expectation zero, then for every $n \geqslant 1$

$$c\int_\Omega \Phi\left([\sum_{k=1}^n x_k^2]^{\frac{1}{2}} \right) \leqslant \int_\Omega \Phi\left(|\sum_{k=1}^n x_k| \right) \leqslant C\int_\Omega \Phi\left([\sum_{k=1}^n x_k^2]^{\frac{1}{2}} \right) \tag{2.8.9}$$

By Theorem (2.5.4), the space $L^\Phi[0,1]$ embeds complementably in X_T^Φ. Application of Theorem (2.7.1) implies that **A** or **B** below is true:

A. There is some n such that $L^\Phi[0,1]$ is isomorphic to a complemented subspace of B_n.

B. there is a block basic sequence (b_r) of the B_n's which is equivalent to the Haar system of $L^\Phi[0,1]$.

Assume (A): It is easily seen that B_n is isomorphic to $X_{T_n}^\Phi \bigoplus X_{T_n}^\Phi$. So by another application of Theorem (2.7.1), $L^\Phi[0,1]$ should embed complementably in $X_{T_n}^\Phi$. This however is impossible by induction hypothesis since $\circ[T_n] < \circ[T]$.
Assume (B): A block basic sequence of the B_n's is a sequence of probabilistically independent functions which is equivalent to the Haar system of $L^\Phi[0,1]$. By Proposition (2.5.3), we have that $L^\Phi[0,1]$ is isomorphic to a modular sequence space $\ell_{(\varphi_n)}$. However, Proposition (2.4.5) and Theorem (2.4.6) imply that this space does not contain an $L^\Phi[0,1]$ copy. This contradiction concludes the proof. □

Let $\mathcal{T}$ be the set of all trees on $\mathbb{N}$ which is a closed subset of the Cantor space $\Delta = 2^{\mathbb{N}^{<\mathbb{N}}}$. In addition, we denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure for more about the application of descriptive set theory in the geometry of Banach spaces see e.g., [Bos02], or [AGR03].

Lemma 2.8.6. *Suppose $\psi: \mathcal{T} \mapsto \mathcal{SE}$ is a Borel map, such that if T is well founded tree and S is a tree and $\psi(T) \cong \psi(S)$, then the tree S is well founded. Then there are uncountably many mutually non-isomorphic members in the class $\{\psi(T); T \text{ is well founded tree}\}$.*

Proof. Assume by contradiction that the number of the non-isomorphic members in the class

$$\{\psi(T);\ T \ \textit{is well founded tree}\}$$

is countable, then there exists a countable sequence of well founded trees $(T_i)_{i=1}^\infty$ such that for any well founded tree T there exists i such that $\psi(T)$ is isomorphic to $\psi(T_i)$.
Consider $B_i = \{X \in \mathcal{SE};\ X \cong \psi(T_i)\}$, then B_i is an analytic subset of $\mathcal{SE}$ for all $i \geqslant 1$ because of the analyticity of the isomorphism relation. Moreover, since ψ is Borel map, then $A_i = \{T \in \mathcal{T};\ \psi(T) \cong \psi(T_i)\}$ is analytic subset of $\mathcal{T}$ for all $i \geqslant 1$.
From our hypothesis we get that $\{T;\ T \text{ is well founded tree}\} = \cup_{i\geqslant 1} A_i$ is analytic which is a contradiction. □

Lemma 2.8.7. *The map $\psi: \mathcal{T} \mapsto \mathcal{SE}$ defined by $\psi(T) = X_T^\Phi$ is Borel.*

Proof. Let U be an open set of $C(\Delta)$ and $(\Gamma_i)_{i=1}^\infty$ be a sequence of all the finite branches of $\mathcal{C}$. It is sufficient to prove that $\mathcal{B} = \{T \in \mathcal{T}; \psi(T) \cap U \neq \emptyset\}$ is Borel. It is clear that $\psi(T) \cap U \neq \emptyset$ if and only if there exists $\underline{\lambda} = (\lambda_i)_{i=1}^n \in \mathbb{Q}^{<\mathbb{N}}$ such that $\sum_{i=0}^n \lambda_i w_{\Gamma_i} \in U$ and $\lambda_i = 0$ when $\Gamma_i \not\subset T$.
Let $\Lambda = \{\underline{\lambda} \in \mathbb{Q}^{<\mathbb{N}}; \sum_i \lambda_i w_{\Gamma_i} \in U\}$ and for $\underline{\lambda} \in \mathbb{Q}^{<\mathbb{N}}$ set $\operatorname{supp}(\underline{\lambda}) = \{i \in \mathbb{N}; \lambda_i \neq 0\}$. Then

$$\mathcal{B} = \bigcup_{\underline{\lambda}\in\Lambda} \bigcap_{i \in supp(\underline{\lambda})} \{T \in \mathcal{T}; \Gamma_i \subset T\}. \tag{2.8.10}$$

Since $\{T \in \mathcal{T}; \Gamma_i \subset T\} = \bigcap_{c \in \Gamma_i} \{T \in \mathcal{T}; c \in T\}$ is Borel (because for $c \in \mathcal{C}$ the set $\{T \in \mathcal{T}; c \in T\}$ is clopen subset in $\mathcal{T}$), then $\mathcal{B}$ is Borel. □

We recall that in [JMST79, p. 235], it is shown that the Orlicz function $\Phi(t) = t^2 \exp(f_0(\log(t)))$, where $f_0(u) = \sum_{k=1}^\infty (1 - \cos \frac{\pi u}{2^k})$ is such that the associated Orlicz function space $L^\Phi[0,1]$ and $L^\Phi[0,\infty)$ are isomorphic. Moreover, this space is $(2-\epsilon)$-convex and $(2+\epsilon)$-concave for all $\epsilon > 0$. Hence, the Boyd indices satisfy $\alpha_\Phi = \beta_\Phi = \frac{1}{2}$. In addition, the space $L^\Phi[0,1]$ is not isomorphic to $L^2[0,1]$. Also, Orlicz function spaces are constructed in [HP86] and [HR89] which do not contain any complemented copy of ℓ^p for $p \geqslant 1$. Thus, the next corollary is not a straightforward consequence of [BRS81].

Corollary 2.8.8. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0,1]$.*

Proof. Let ψ be the Borel map defined by $\psi(T) = X_T^\Phi$. Now, let T be a well founded tree and T_0 be a tree such that $X_{T_0}^\Phi$ is isomorphic to a subspace of X_T^Φ, then Theorem (2.8.5) implies that T_0 is well founded. By Theorem (2.8.3), the spaces X_T^Φ are complemented in $L^\Phi(G)$. Hence, there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0,1]$ by Lemma (2.8.6). □

It was mentioned before that the set of all well founded trees is co-analytic non Borel and so the set of all trees which are not well founded (ill founded) is analytic non-Borel. Following [Bos02], if X is a separable Banach space, then $< X >$ denotes the equivalence class $\{Y \in \mathcal{SE};\ Y \simeq X\}$. We have the following result.

Corollary 2.8.9. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $< L^\Phi[0,1] >$ is analytic non Borel.*

Proof. Since the isomorphism relation $\{(X,Y);\ X \simeq Y\}$ is analytic in $\mathcal{SE}^2$ by [Bos02, Theorem 2.3], then the class $< L^\Phi[0,1] >$ is analytic. Moreover, since ψ is Borel and $\psi^{-1}(< L^\Phi[0,1] >) = \{T;\ T \textit{ is ill founded}\}$ by Theorem (2.8.5), then the class $< L^\Phi[0,1] >$ is non-Borel. □

In [Bos02], it has been shown that $< \ell^2 >$ is Borel. It is unknown whether this condition characterizes the Hilbert space, and thus we recall [Bos02, Problem 2.9]: Let X be a separable Banach space whose isomorphism class $< X >$ is Borel. Is X isomorphic to ℓ^2? A special case of this problem seems to be of particular importance, namely: Is the isomorphism class $< c_0 >$ of c_0 Borel? For more about this question and analytic sets of Banach spaces see [God10].

Appendix A

The first article

This thesis contains three articles that the author published during the preparation of her Ph.D. The first article is [Ghaa].

The descriptive complexity of the family of Banach spaces with the bounded approximation property

keywords: Borel set; analytic set; a FDD; bounded approximation property; comeager

Abstract. We show that the set of all separable Banach spaces that have the bounded approximation property (BAP) is a Borel subset of the set of all closed subspaces of $C(\Delta)$, where Δ is the Cantor set, equipped with the standard Effros-Borel structure. Also, we prove that if X is a separable Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^{\infty}$ and $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u \ has\ a\ FDD\}$ is comeager in Δ.

A.1 Introduction

Let $C(\Delta)$ be the space of continuous functions on the Cantor space Δ. It is well-known that $C(\Delta)$ is isometrically isomorphic universal for all separable Banach spaces. We denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure. In [Bos02], B. Bossard considered the topological complexity of the isomorphism relation and many other subsets of $\mathcal{SE}$. We recall that the Banach space X has the λ-Bounded Approximation Property (λ-BAP), $\lambda \geqslant 1$, if for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \longrightarrow X$ with $\|T\| \leqslant \lambda$ and $\|T(x) - x\| < \epsilon$ for every $x \in K$.

The main result of this note asserts that the set of all separable Banach spaces that have the BAP is a Borel subset of $\mathcal{SE}$. Furthermore, we show that if X is a separable

Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^{\infty}$ and $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u\ has\ a\ FDD\}$ is comeager in Δ.

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A.2 Main Results

Here is the first result of this note.

Lemma A.2.1. *Suppose $(x_n)_{n=1}^{\infty}$ is a dense sequence in a Banach space X and $\lambda > 1$. Then X has the λ-bounded approximation property if and only if*

$$\forall K\ \forall \epsilon > 0\ \exists \lambda' < \lambda\ \exists R\ \forall N\ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R\ \forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{i=1}^{R} \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{A.2.1}$$

$$\forall i \leqslant K\ \|x_i - [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \epsilon \tag{A.2.2}$$

where $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda'$ vary over $\mathbb{Q}$.

Proof. Indeed, suppose X has the λ-bounded approximation property and let K and $\epsilon > 0$ be given. Then there is a finite rank operator $T : X \mapsto X$ of norm $< \lambda$ so that $\|z - T(z)\| < \epsilon$ for all z in the compact set $\{x_1, \cdots, x_K\}$. By perturbing T, we may suppose that T maps into the finite-dimensional subspace $[x_1, \cdots, x_R]$ for some R. Pick some $\|T\| < \lambda' < \lambda$. Then, for every N, we may perturb T slightly so that $\|T\| < \lambda'$ and that $T(x_i)$ belongs to the $\mathbb{Q}$-linear span of the $(x_j)_{j \leqslant R}$ for all $i \leqslant N$. Define now $\sigma_i \in \mathbb{Q}^R$ by $T(x_i) = \sum\limits_{j=1}^{R} \sigma_i(j) x_j$ and let $\lambda' = \|T\|$. Then the two inequalities above hold for all $\alpha_1, \cdots, \alpha_N$ and $i \leqslant K$.

Conversely, suppose that the above criterion holds and that $C \subseteq X$ is compact and $\epsilon' > 0$. Pick a rational $\frac{\epsilon'}{3\lambda} > \epsilon > 0$ and a K so that every point of C is within ϵ of some $x_i,\ i \leqslant K$. So let λ' and R be given as above. Then, for every N and all $i \leqslant N$, define $y_i^N = \sum\limits_{j=1}^{R} \sigma_i(j) x_j \in [x_1, \cdots, x_R]$, where the σ_i are given depending on N. We have that

$$\|\sum_{i=1}^{N} \alpha_i y_i^N\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|$$

for all $\alpha_i \in \mathbb{Q}$, and

$$\|x_i - y_i^N\| < \epsilon$$

for all $i \leqslant K$. In particular, for every i, the sequence $(y_i^N)_{N=i}^{\infty}$ is contained in a bounded set in a finite-dimensional space. So by a diagonal procedure, we may find some subsequence (N_l) so that $y_i = \lim\limits_{l \to \infty} y_i^{N_l}$ exists for all i. By consequence

$$\|\sum_{i=1}^{\infty} \alpha_i y_i\| \leqslant \lambda' \cdot \|\sum_{i=1}^{\infty} \alpha_i x_i\|$$

for all $\alpha_i \in \mathbb{Q}$, and

$$\|x_i - y_i\| \leqslant \epsilon$$

for all $i \leqslant K$.

Now, since the x_i are dense in X, there is a uniquely defined bounded linear operator $T : X \mapsto [x_1, \cdots, x_R]$ satisfying $T(x_i) = y_i$. Moreover, $\|T\| \leqslant \lambda' < \lambda$ and $\|x_i - T(x_i)\| \leqslant \epsilon$ for all $i \leqslant K$. It follows that $\|z - T(z)\| < \epsilon'$ for all $z \in C$. ☐

Note that in the next result, we may replace "BAP" by "Lipschitz BAP" since these two notions are equivalent, (see [GK03, Theorem(5.3)]).

Theorem A.2.2. *The set of all separable Banach spaces that have the BAP is a Borel subset of $\mathcal{SE}$.*

Proof. Let $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda'$ vary over $\mathbb{Q}$, then we consider the set $E_{K,\epsilon,\lambda',R,N,\sigma,\alpha} \subseteq C(\Delta)^{\mathbb{N}}$ such that:

$$E_{K,\epsilon,\lambda',R,N,\sigma,\alpha} = \{(x_n)_{n=1}^{\infty} \in C(\Delta)^{\mathbb{N}};\ (A.2.1)\ \textit{and}\ (A.2.2)\ \textit{hold}\}$$

This set is closed in $C(\Delta)^{\mathbb{N}}$. Therefore, for $\lambda \in \mathbb{Q}$

$$E_\lambda = \bigcap_K \bigcap_\epsilon \bigcup_{\lambda'<\lambda} \bigcup_R \bigcap_N \bigcup_{\sigma\in(\mathbb{Q}^R)^N} \bigcap_{\alpha\in\mathbb{Q}^N} E_{K,\epsilon,\lambda',R,N,\sigma,\alpha}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. Moreover, the set

$$E = \bigcup_{\lambda\in\mathbb{Q}} E_\lambda$$

is also Borel.

There is a Borel map $d : \mathcal{SE} \longrightarrow C(\Delta)^{\mathbb{N}}$ such that $\overline{d(X)} = X$, by [Kec95, Theorem(12.13)]. Moreover, the previous Lemma implies that

$$X \textit{ has the BAP} \iff d(X) \in E$$

Therefore, $\{X \in \mathcal{SE};\ X \textit{ has the BAP}\}$ is a Borel subset of $\mathcal{SE}$. ☐

Our argument shows:

Proposition A.2.3. *The map $\psi : \{X \in \mathcal{SE};\ X \textit{ has the BAP}\} \longrightarrow [1,\infty[$, defined by $\psi(X) = inf\{\lambda;\ X \textit{ has the } \lambda - BAP\}$, is Borel.*

Proof. Let $\lambda' \in \mathbb{Q}$, and $G_{\lambda'} = \bigcap_K \bigcap_\epsilon \bigcup_R \bigcap_N \bigcup_{\sigma\in(\mathbb{Q}^R)^N} \bigcap_{\alpha\in\mathbb{Q}^N} E_{K,\epsilon,\lambda',R,N,\sigma,\alpha}$.

$$\psi(X) \leqslant \lambda' \iff d(X) \in G_{\lambda'}$$

and so

$$\{X \in \mathcal{SE};\ \psi(X) \leqslant \lambda'\}$$

is a Borel set. Thus, for all $\lambda' \in \mathbb{Q}$, $\psi^{-1}([1,\ \lambda'])$ is Borel. Therefore, ψ is a Borel map. ☐

Proposition A.2.4. *The set of all Banach spaces with a Schauder basis is an analytic subset of $\mathcal{SE}$.*

Proof. Let $R \in \mathbb{Q}$, $\alpha = (a_i) \in \mathbb{Q}^{<\mathbb{N}}$ and $n,\ m \in \mathbb{N}$, with $n < m \leqslant |\alpha|$. The set

$$A_{R,\alpha,m,n} = \{(x_i) \in C(\Delta)^{\mathbb{N}};\ \| \sum_{j=1}^{n} a_i\, x_i \| \leqslant R\, \| \sum_{j=1}^{m} a_i\, x_i \| \}.$$

is closed in $C(\Delta)$. Therefore,

$$A = \bigcup_{R} \bigcap_{\alpha} \bigcap_{m \leqslant |\alpha|} \bigcap_{n \leqslant m} A_{R,a,m,n}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. By Lemma(2.6.ii) in [Bos02], the set:

$$\Omega = \{((x_i), X);\ X = \overline{span}(x_i)\ and\ (x_i) \in A\}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}} \times \mathcal{SE}$. Let $\pi :\ C(\Delta)^{\mathbb{N}} \times \mathcal{SE} \longrightarrow \mathcal{SE}$ be the canonical projection. Then, $\pi((A \times \mathcal{SE}) \bigcap \Omega) = \{X \in \mathcal{SE};\ X\ has\ a\ basis\}$ is an analytic subset of $\mathcal{SE}$. □

It is unknown if the set of all separable Banach spaces with a basis is Borel or not. If it is non Borel set, then this will give a non trivial result that the set of all separable Banach spaces with the BAP and without a basis is co-analytic non-Borel. It has been shown by S. J. Szarek in [Sza87] that this set is not empty.

Some works have been done on the relation between Baire Category and families of subspaces of a Banach space with a Schauder basis, (see e.g. [FG12]). In order to state the next theorem we recall that a fundamental and total biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called a Markushevich basis (M-basis) for $(X, \|.\|)$. Furthermore, a biorthogonal system $\{e_i, e_i^*\}_{i=1}^{\infty}$ in $X \times X^*$ is called λ- norming if $|||x||| = sup\{|x^*(x)|;\ x^* \in B_{X^*} \bigcap \overline{span}^{||\cdot||}\{e_i^*\}_{i=1}^{\infty}\}$, $x \in X$, is a norm satisfying $\lambda||x|| \leqslant |||x|||$ for some $0 < \lambda \leqslant 1$, (see [?]). Also, any separable Banach space has a bounded norming M-basis, (see [Ter94]). A separable Banach space X has a finite dimensional decomposition (a FDD) $\{X_n\}_{n=1}^{\infty}$, where X_n's are finite dimensional subspaces of X, if for every $x \in X$ there exists a unique sequence (x_n) with $x_n \in X_n$, $n \in \mathbb{N}$, such that $x = \sum\limits_{n=1}^{\infty} x_n$.

Theorem A.2.5. *Let X be a separable Banach space with a norming M-basis $\{e_i, e_i^*\}_{i=1}^{\infty}$. If we let $E_u = \overline{span}\{e_i;\ i \in u\}$ for $u \in \Delta$, then the set $\{u \in \Delta;\ E_u\ has\ a\ FDD\}$ is comeager in the Cantor space Δ.*

Proof. We may and do assume that the M-basis is 1-norming. We will consider $\{I_j\}_{j=0}^{\infty}$ successive intervals of $\mathbb{N}$ where, $I_0 = \emptyset$ and for $j \geqslant 1$ we will write $I_j = F_j \cup G_j$, such that $F_j \cap G_j = \emptyset$, and $F_j = [min\ I_j,\ min\ I_j + \rho_j]$, such that $\rho_j \in \mathbb{N}$ and $\rho_j \leqslant (max\ I_j - min\ I_j)$. We can construct the sequence $\{I_j\}_{j=0}^{\infty}$ inductively satisfying the following property:

for $n \in \mathbb{N}$ and $x \in span\{e_i;\ i \in \bigcup\limits_{j=0}^{n-1} I_j \bigcup F_n\}$

$$\| x \| \leq (1 + \frac{1}{n}) \sup\{|x^*(x)|;\ \|x^*\| = 1,\ and\ x^* \in span\{e_i^*;\ i \in \bigcup_{j=0}^{n} I_j\}\} \tag{A.2.3}$$

Indeed, let $H_n = span\{e_i;\ i \in \bigcup\limits_{j=0}^{n-1} I_j \bigcup F_n\}$, then for any $\epsilon > 0$, S_{H_n} has an $\frac{\varepsilon}{2} - net$ finite set $A_n = \{y_{t,n}\}_{t=1}^{m_n}$ [such that (m_n) is an increasing sequence in $\mathbb{N}$ and $m_n = |A_n|$]. Moreover, we can choose for every $y_{t,n} \in A_n$, a functional map $y^*_{t,n} \in S_{X^*} \bigcap span\{e_i^*;\ i \geq 1\}$ where

$$|y^*_{t,n}(y_{t,n})| > 1 - \tfrac{\epsilon}{2}$$

Therefore, for any $x \in S_{H_n}$, there exists $y_{t,n} \in A_n$ such that:

$$||y_{t,n} - x|| < \tfrac{\epsilon}{2}$$

Then,

$$|y^*_{t,n}(y_{t,n}) - y^*_{t,n}(x)| < \frac{\epsilon}{2} \tag{A.2.4}$$

$$|y^*_{t,n}(x)| > 1 - \epsilon \tag{A.2.5}$$

Therefore,

$$||x|| < (1-\epsilon)^{-1}|y^*_{t,n}(x)|, \ \forall x \in H_n \tag{A.2.6}$$

In particular, for $\epsilon = \frac{1}{n+1}$ we have the functional maps $\{y^*_{t,n}\}_{t=1}^{m_n}$ such that (A.2.6) holds. In order to complete our construction, let $supp(y^*_{t,n}) = \{i;\ y^*_{t,n}(e_i) \neq 0\}$, then $\bigcup\limits_{t=1}^{m_n} supp(y^*_{t,n}) = B_n$ is a finite set. Now, we can choose G_n to be a finite set such that

$$B_n \subseteq (\bigcup_{j=0}^{n-1} I_j \bigcup F_n \bigcup G_n) = \bigcup_{j=0}^{n} I_j$$

Suppose $u \in \Delta$ and $\{F_{n_k}\}$ is an infinite subsequence such that $u \bigcap I_{n_k} = F_{n_k}$ for all $k \in \mathbb{N}$. If $x \in \overline{span}\{e_i;\ i \in u\}$, then $x \in \{e^*_i;\ i \in G_{n_k}\}^{\perp}$, for all $k \in \mathbb{N}$. And so, $x = x_{n_k} + y$, where $x_{n_k} \in span\{e_i;\ i \in \Gamma_{n_k}\}$ such that $\Gamma_{n_k} = u \bigcap (\bigcup\limits_{j=0}^{n_k-1} I_j \bigcup F_{n_k})$, and $y \in span\{e_i;\ i \in u \backslash \Gamma_{n_k}\}$. Define for every k a linear projection $P_{n_k} :\ span\{e_i;\ i \in u\} \longrightarrow span\{e_i;\ i \in u\}$, such that $P_{n_k}(x) = \sum\limits_{i \in \Gamma_{n_k}} e^*_i(x) e_i$. Then

$$\begin{aligned}
\| P_{n_k}(x) \| &\leq (1 + \frac{1}{n_k}) \sup\{|x^*(P_{n_k}(x))|;\ \|x^*\| = 1,\ and\ x^* \in span\{e^*_i;\ i \in \bigcup_{j=0}^{n_k} I_j\}\} \\
&\leq (1 + \frac{1}{n_k}) \sup\{|x^*(x)|;\ \|x^*\| = 1,\ and\ x^* \in span\{e^*_i;\ i \in \bigcup_{j=0}^{n_k} I_j\}\} \\
&\leq (1 + \frac{1}{n_k}) \| x \|
\end{aligned}$$

Therefore, $\{P_{n_k}\}_{k=1}^{\infty}$ is a sequence of bounded linear projections such that $\sup\limits_k \| P_{n_k} \| < \infty$, and $P_{n_{k_2}} P_{n_{k_1}} = P_{\min(n_{k_1}, n_{k_2})}$. Hence, these projections determine a finite dimensional decomposition (a FDD) of the subspace $E_u = \overline{span}\{e_i;\ i \in u\}$ by putting $E^1_u = P_{n_1}(E_u)$ and $E^k_u = (P_{n_k} - P_{n_{k-1}})(E_u)$, for $k > 1$.
Therefor, E_u has a FDD if $\{n;\ u \bigcap I_n = F_n\}$ is infinite, thus $\{u \in \Delta;\ E_n\ has\ a\ FDD\}$ is comeager, by [FG12, Lemma 2.3]. □

Note that since every space with a FDD has the BAP. Theorem (A.2.5) shows that $\{u \in \Delta;\ E_n\ has\ the\ BAP\}$ is comeager in Δ. We recall that an example of a separable Banach space with the BAP but without a FDD has been constructed by C. Read, (see [CK91]).

Appendix B

The second article

This is the second article which is a part of Chapter one, (see [Gha15]).

The descriptive complexity of the family of Banach spaces with the π-property

keywords: π_λ-property; analytic set; Borel set; a FDD; bounded approximation property

Abstract. We show that the set of all separable Banach spaces that have the $\pi-$property is a Borel subset of the set of all closed subspaces of $C(\Delta)$, where Δ is the Cantor set, equipped with the standard Effros-Borel structure. We show that if $\alpha < \omega_1$, the set of spaces with Szlenk index at most α which have a shrinking FDD is Borel.

B.1 Introduction

Let $C(\Delta)$ be the space of continuous functions on the Cantor space Δ. It is well-known that $C(\Delta)$ is isometrically universal for all separable Banach spaces. We denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure. In [Bos02], B. Bossard considered the topological complexity of the isomorphism relation and of many subsets of $\mathcal{SE}$. In addition, it has been shown that the set of all separable Banach spaces that have the bounded approximation property (BAP) is a Borel subset of $\mathcal{SE}$, and that the set of all separable Banach spaces that have the metric approximation property (MAP) is also Borel [Ghaa].
We recall that the Banach space X has the $\pi_\lambda-$property if there is a net of finite rank projections (S_α) on X converging strongly to the identity on X with $\limsup_\alpha \|S_\alpha\| \leqslant \lambda$. (see [Cas01]). We say that the Banach space X has the $\pi-$property if it has the $\pi_\lambda-$property for some $\lambda \geq 1$.

In this note we show that the set of all separable Banach spaces that have the $\pi-$property is a Borel subset of $\mathcal{SE}$. This bears some consequences on the complexity of the class of spaces with finite dimensional decompositions. For instance, we show that in the set of spaces whose Szlenk index is bounded by some countable ordinal, the subset consisting of spaces which have a shrinking finite dimensional decomposition is Borel.

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B.2 main results

Here is the main technical lemma.

Lemma B.2.1. *Suppose $(x_n)_{n=1}^{\infty}$ is a dense sequence in a Banach space X and $\lambda > 1$. Then X has the π-property if and only if*
$\forall c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}\ \forall K\ \forall \epsilon > 0\ \forall \lambda' > \lambda\ \exists R\ \forall N \geqslant R\ \exists \sigma_1, \cdots, \sigma_N \in \mathbb{Q}^R\ \forall \alpha_1, \cdots, \alpha_N \in \mathbb{Q}$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{B.2.1}$$

$$\forall i \leqslant K,\ \|x_i - [\sum_{j=1}^{R} \sigma_i(j) x_j]\| \leqslant \epsilon \tag{B.2.2}$$

$$\|\sum_{i=1}^{N} \alpha_i [\sum_{j=1}^{R} \sigma_i(j) x_j] - \sum_{i=1}^{N} \alpha_i [\sum_{t=1}^{R} [\sum_{j=1}^{R} \sigma_i(j) \sigma_j(t)] x_t]\| \leqslant c \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{B.2.3}$$

where $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda'$ vary over $\mathbb{Q}$.

Proof. Indeed, suppose X has the π_λ -property, then there exists a sequence (P_n) of finite rank projections such that $\|P_n\| < \lambda$, for all n and P_n converges strongly to the identity. By perturbing P_n, we may suppose that P_n maps into the finite-dimensional subspace $[x_1, \cdots, x_{R_n}]$ for some R_n in $\mathbb{N}$ but we still have (B.2.2) and $\|P_n\| < \lambda$. Pick $\lambda' \in \mathbb{Q}$ with $\|P_n\| < \lambda' < \lambda$. Then, for every N, we may perturb P_n slightly so that $\|P_n\| < \lambda'$ and $P_n(x_i)$ belongs to the $\mathbb{Q}$-linear span of the x_j for all $i \leqslant N$, such that (B.2.1), (B.2.2) and (B.2.3) still hold, and $P_n^2(x_i) = P_n(x_i)$. Define now $(\sigma_i^{(n)}) \in (\mathbb{Q}^{R_n})^N$, such that $P_n(x_i) = \sum_{j=1}^{R_n} \sigma_i^{(n)}(j) x_j$. Since $P_n^2(x_i) = \sum_{t=1}^{R_n} [\sum_{j=1}^{R_n} \sigma_i(j) \sigma_j(t)] x_t$, the three inequalities hold for all $\alpha_1, \cdots, \alpha_N \in \mathbb{Q}$, and $i \leqslant K$.

Conversely, suppose that the above criterion holds and that $\epsilon' > 0$. Pick a rational $\frac{\epsilon'}{3\lambda} > \epsilon > 0$ and a K. So let λ' and R be given as above. Then for every N and $i \leqslant N$, define $y_i^N = \sum_{j=1}^{R} \sigma_i(j) x_j$, and $z_i^N = \sum_{t=1}^{R} [\sum_{j=1}^{R} \sigma_i(j) \sigma_j(t)] x_t$ in $[x_1, \cdots x_R]$, where the σ_i are given depending on N. We have that

$$\|\sum_{i=1}^{N} \alpha_i y_i^N\| \leqslant \lambda' \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{B.2.4}$$

for all $\alpha_i \in \mathbb{Q}$,

$$\|x_i - y_i^N\| \leqslant \epsilon \tag{B.2.5}$$

for all $i \leqslant K$, and

$$\|\sum_{i=1}^{N} \alpha_i y_i^N - \sum_{i=1}^{N} \alpha_i z_i^N\| \leqslant c \cdot \|\sum_{i=1}^{N} \alpha_i x_i\|. \tag{B.2.6}$$

for all $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$. In particular, for every i, the sequences $(y_i^N)_{N=i}^{\infty}$, and $(z_i^N)_{N=i}^{\infty}$ are contained in a bounded set in a finite-dimensional space. So by a diagonal procedure, we may find some subsequence (N_l) so that $y_i = \underset{l \to \infty}{lim} y_i^{N_l}$ and $z_i = \underset{l \to \infty}{lim} z_i^{N_l}$ exists for all i. By consequence

$$\|\sum_{i=1}^{\infty} \alpha_i y_i\| \leqslant \lambda' \cdot \|\sum_{i=1}^{\infty} \alpha_i x_i\|. \tag{B.2.7}$$

for all $\alpha_i \in \mathbb{Q}$,

$$\|x_i - y_i\| \leqslant \epsilon \tag{B.2.8}$$

for all $i \leqslant K$, and

$$\|\sum_{i=1}^{\infty} \alpha_i y_i - \sum_{i=1}^{\infty} \alpha_i z_i\| \leqslant c \cdot \|\sum_{i=1}^{\infty} \alpha_i x_i\|. \tag{B.2.9}$$

for all $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$.

Now, since the x_i are dense in X, there are uniquely defined bounded linear operators $T_{K,\epsilon} : X \mapsto [x_1, \cdots, x_R]$ satisfying $T_{K,\epsilon}(x_i) = y_i$ and then $T^2_{K,\epsilon} : X \mapsto [x_1, \cdots, x_R]$ satisfies $T^2_{K,\epsilon}(x_i) = z_i$ such that $\|T_{K,\epsilon}\| \leqslant \lambda' < \lambda$ and $\|x_i - T_{K,\epsilon}(x_i)\| \leqslant \epsilon$ for all $i \leqslant K$. Let $K \to \infty$ and $\epsilon \to 0$, then $T_{K,\epsilon}(x_i) \to x_i$ for all $x_i \in (x_i)$ strongly. Since (x_i) is a dense sequence in X and the operators $T_{K,\epsilon}$ are uniformly bounded, then $T_{K,\epsilon}(x) \to x$ for all $x \in X$ strongly. Also, $\underset{n \to \infty}{lim}\, sup \|T_{K,\epsilon} - T^2_{K,\epsilon}\| < \frac{1}{4}$. Therefore, by [CK91, Theorem(3.7)], X has the $\pi_{\lambda+1}$-property. □

Theorem B.2.2. *The set of all separable Banach spaces that have the π-property is a Borel subset of $\mathcal{SE}$.*

Proof. Let $K,\ R,\ N$ vary over $\mathbb{N}$ and $\epsilon,\ \lambda'$ vary over $\mathbb{Q}$. Let also $c \in (0, \frac{1}{4}) \bigcap \mathbb{Q}$, $\sigma \in (\mathbb{Q}^R)^N$, and $\alpha \in \mathbb{Q}^N$, then we consider the set $E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha} \subseteq C(\Delta)^{\mathbb{N}}$ such that:

$$E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha} = \{(x_n)_{n=1}^{\infty} \in C(\Delta)^{\mathbb{N}};\ (1.5.2)\ (1.5.3)\ \textit{and}\ (1.5.4)\ \textit{hold}\}$$

This set is closed in $C(\Delta)^{\mathbb{N}}$. Therefore, for $\lambda \in \mathbb{R}$

$$E_\lambda = \bigcup_{c} \bigcap_{K} \bigcap_{\epsilon} \bigcup_{\lambda' < \lambda} \bigcup_{R} \bigcap_{N} \bigcup_{\sigma \in (\mathbb{Q}^R)^N} \bigcap_{\alpha \in \mathbb{Q}^N} E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha}$$

is a Borel subset of $C(\Delta)^{\mathbb{N}}$. Moreover, the set

$$E = \bigcup_{\lambda \in \mathbb{Q}} E_\lambda$$

is also Borel.
There is a Borel map $d : \mathcal{SE} \longrightarrow C(\Delta)^{\mathbb{N}}$ such that $\overline{d(X)} = X$, by [Kec95, Theorem(12.13)]. Moreover, the previous Lemma implies that

$$X\ \textit{has the}\ \pi - \textit{property} \iff d(X) \in E.$$

Therefore, $\{X \in \mathcal{SE};\ X\ \textit{has the}\ \pi - \textit{property}\}$ is a Borel subset of $\mathcal{SE}$. □

We will now prove that this result implies, with some work, that in some natural classes the existence of a finite-dimensional decomposition happens to be a Borel condition. We first consider the class of reflexive spaces. The commuting bounded approximation property (CBAP) implies the bounded approximation property (BAP) by the definition of the CBAP. By Grothendieck's theorem (see [LT77, Theorem 1.e.15]) the BAP and the metric approximation property (MAP) are equivalent for reflexive Banach spaces. In addition, [CK91, Theorem 2.4] implies that for any reflexive Banach space the CBAP is equivalent to MAP. For the set R of all separable reflexive Banach spaces, Theorem (B.2.2), [Ghaa, Theorem 2.2] and [Cas01, Theorem 6.3] imply that there exists a Borel subset $B = \{X \in \mathcal{SE};\ X\ has\ the\ MAP\ and\ the\ \pi - property\}$ such that $\{X \in R; X\ has\ a\ FDD\} = B \cap R$.

We will extend this simple observation to some classes of non reflexive spaces. The following result has been proved in [Joh72]. The proof below follows the lines of [GS88].

Proposition B.2.3. *Let X be a Banach space with separable dual. If X has the MAP for all equivalent norms then X^* has the MAP.*

Proof. Since X^* is separable, there is an equivalent Fréchet differentiable norm on X. If $\|.\|_X$ is a Fréchet differentiable norm and $x \in S_X$, there exists a unique $x^* \in S_{X^*}$ such that $x^*(x) = 1$, and x^* is a strongly exposed point of B_{X^*}. Since by assumption X equipped with this norm has the MAP, there exists an approximating sequence (T_n) with $\|T_n\| \leqslant 1$, and then for all $x^* \in X^*$ we have $T_n^*(x^*) \overset{w^*}{\to} x^*$. For all $x^* \in X^*$ which attains its norm we have $\|T_n^*(x^*) - x^*\|_{X^*} \to 0$. Bishop-Phelps theorem yields that for all $x^* \in X^*$, $\|T_n^*(x^*) - x^*\|_{X^*} \to 0$. Therefore, X^* has the MAP. □

The set SD of all Banach spaces with separable dual spaces is coanalytic in $\mathcal{SE}$ and the Szlenk index Sz is a coanalytic rank on SD (see [Bos02, Corollary (3.3) and Theorem (4.11)]). In particular, the set $S_\alpha = \{X \in \mathcal{SE};\ Sz(X) \leqslant \alpha\}$ is Borel in $\mathcal{SE}$ (see [Kec95]). In this Borel set, the following holds.

Theorem B.2.4. *The set of all separable Banach spaces in S_α that have a shrinking FDD is Borel in $\mathcal{SE}$.*

Proof. Indeed, by [Dod10, Theorem 1], we have that

$$S_\alpha^* = \{Y \in \mathcal{SE};\ \exists X \in S_\alpha\ with\ Y \simeq X^*\}$$

is analytic. Since the set $\{Y \in \mathcal{SE};\ Y\ has\ the\ BAP\}$ is Borel by [Ghaa, Theorem 2.2], then

$$G_\alpha^* = \{Y \in \mathcal{SE};\ \exists X \in S_\alpha\ with\ Y \simeq X^*\ and\ Y\ has\ the\ BAP\}$$

is analytic. By [Dod10, Proposition 7], we have that

$$G_\alpha = \{X \in S_\alpha;\ \exists Y \in G_\alpha^*,\ with\ Y \simeq X^*\}$$

is analytic.
Since $\{(X,Z); Z \simeq X\}$ is analytic in $\mathcal{SE} \times \mathcal{SE}$ ([Bos02, Theorem 2.3]) and $\{Z;\ Z\ fails\ the\ MAP\}$ is Borel [Ghaa], the set $\{(X,Z); Z \simeq X,\ Z\ fails\ the\ MAP\}$ is analytic, thus its canonical

projection $\{X \in \mathcal{SE};\ \exists Z \in \mathcal{SE}; Z \simeq X,\ Z\ fails\ the\ MAP\}$ is analytic. Now, Proposition (B.2.3) implies that the set

$$\begin{aligned} H_\alpha =& \{X \in S_\alpha;\ X^*\ fails\ the\ AP\} \\ =& \{X \in S_\alpha;\ \exists Z\ with\ Z \simeq X\ and\ Z\ fails\ the\ MAP\} \end{aligned}$$

is analytic. Since $S_\alpha \setminus H_\alpha = G_\alpha$ and both G_α and H_α are analytic sets in $\mathcal{SE}$, then both are Borel by the separation theorem. Now, [Cas01, Theorem 4.9] implies that $G_\alpha = \{X \in S_\alpha;\ X\ has\ the\ shrinking\ CBAP\}$. Thus,

$$\{X \in S_\alpha;\ X\ has\ a\ shrinking\ FDD\}$$

is Borel by Theorem (B.2.2) and [Cas01, Theorem 6.3]. □

Questions: As seen before, a separable Banach space has CBAP if and only if it has an equivalent norm for which it has MAP. It follows that the set $\{X \in \mathcal{SE}; X\ has\ the\ CBAP\}$ is analytic. It is not clear if it is Borel or not. Also, it is not known if there is a Borel subset B of $\mathcal{SE}$ such that $\{X \in SD;\ X^*\ has\ the\ AP\} = B \cap SD$. This would be an improvement of Theorem (B.2.4). Finally, what happens when we replace FDD by basis is not clear: for instance, the set of all spaces in S_α which have a basis is clearly analytic. Is it Borel?

Appendix C

The third article

This is the third article which is contained in Chapter two (see [Ghab]).

Non-isomorphic complemented subspaces of Orlicz function spaces L^Φ

keywords: Orlicz function space, complemented subspaces, Cantor group, rearrangement invariant function space, well founded tree, analytic, non Borel

Abstract. In this note we show that the number of isomorphism classes of complemented subspaces of a reflexive Orlicz function space $L^\Phi[0,1]$ is uncountable, as soon as $L^\Phi[0,1]$ is not isomorphic to $L^2[0,1]$. Also, we prove that the set of all separable Banach spaces that are isomorphic to such an $L^\Phi[0,1]$ is analytic non Borel. Moreover, by using Boyd interpolation theorem we extend some results on $L^p[0,1]$ spaces to the rearrangement invariant function spaces under natural conditions on their Boyd indices.

C.1 Introduction

Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. Consider the Cantor group $G = \{-1,1\}^{\mathcal{C}}$ equipped with the Haar measure. The dual group is the discrete group formed by Walsh functions $w_F = \prod_{c\in F} r_c$ where F is a finite subset of $\mathcal{C}$ and r_c is the Rademacher function, that is $r_c(x) = x(c)$, $x \in G$. These Walsh functions generate $L^p(G)$ for $1 \le p < \infty$, and the reflexive Orlicz function spaces $L^\Phi(G)$, where Φ is an Orlicz function.

A measurable function f on G only depends on the coordinates $F \subset \mathcal{C}$, provided $f(x) = f(y)$ whenever $x, y \in G$ with $x(c) = y(c)$ for all $c \in F$. A measurable subset S of G depends only on the coordinates $F \subset \mathcal{C}$ provided χ_S does. Moreover, for $F \subset \mathcal{C}$ the sub-σ-algebra $\mathfrak{G}(F)$ contains all measurable subsets of G that depend only on the

F-coordinates. A branch in $\mathcal{C}$ will be a subset of $\mathcal{C}$ consisting of mutually comparable elements. For more the reader is referred to [BRS81], [Bou81] and [DK14].

In [BRS81], the authors considered the subspace $X_{\mathcal{C}}^p$ which is the closed linear span in $L^p(G)$ over all finite branches Γ in $\mathcal{C}$ of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. In addition, they proved that $X_{\mathcal{C}}^p$ is complemented in $L^p(G)$ and isomorphic to L^p, for $1 < p < \infty$. Moreover, for a tree T on $\mathbb{N}$, the space X_T^p is the closed linear span in $L^p(G)$ over all finite branches Γ in T of all those functions in $L^p(G)$ which depend only on the coordinates of Γ. Hence, X_T^p is a one-complemented subspace of $X_{\mathcal{C}}^p$ by the conditional expectation operator with respect to the sub-σ-algebra $\mathfrak{G}(T)$ which contains all measurable subsets of G that depend only on the T-coordinates. J. Bourgain in [Bou81] showed that the tree T is well founded if and only if the space X_T^p does not contain a copy of L^p, for $1 < p < \infty$ and $p \neq 2$. Consequently, it was shown that if B is a universal separable Banach space for the elements of the class $\{X_T^p; T \textit{ is a well founded tree}\}$, then B contains a copy of L^p. It follows that there are uncountably many mutually non-isomorphic members in this class.

In this note we will show that these results extend to the case of the reflexive Orlicz function spaces $L^{\Phi}[0,1]$, where Φ is an Orlicz function. Moreover, some of the results extend to rearrangement invariant function spaces under some conditions on the Boyd indices.

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C.2 Notations

A rearrangement invariant function space X (r.i. function space) on the interval $I = [0,1]$ or $[0,\infty)$ is a Banach space of equivalence classes of measurable functions on I such that:

(i) $X(I)$ is a Banach lattice with respect to the pointwise order.

(ii) For every automorphism τ of I (i.e., an invertible transformation τ from I onto itself so that, for any measurable subset E of I, $\mu(\tau^{-1}E) = \mu(E)$) and every $f \in X(I)$, also $f(\tau) \in X(I)$ and $\|f(\tau)\| = \|f\|$.

(iii) For $I = [0,1]$ we have $L^\infty([0,1]) \subset X[0,1] \subset L^1([0,1])$ with norm one embeddings. Moreover, for $I = [0,\infty)$ we have $L^1[0,\infty) \bigcap L^\infty[0,\infty) \subset X[0,\infty) \subset L^1[0,\infty) + L^\infty[0,\infty)$ with norm one embeddings.

(iv) $L^\infty([0,1])$ is dense in $X[0,1]$. For $[0,\infty)$, the simple functions with bounded support are dense in $X[0,\infty)$.

For more a bout the definition of the r.i. function space $X(\Omega,\Sigma,\nu)$ on the separable measure space (Ω,Σ,ν) see [LT79, p.114-117].
We recall the definition of the indices introduced by D. Boyd in [Boy69]. If X is a r.i. function space on $[0,1]$, define the dilation mapping D_s, $0 < s < \infty$, by the formula

$$(D_s f)(t) = f(st),\ t \in [0,1], \text{ and } f \in X.$$

In order to make this definition meaningful, the function f is extended to $[0,\infty)$ by $f(u) = 0$ for $u > 1$. Define now the indices

$$\alpha_X = \inf_{0<s<1}\left(\frac{-Log\|D_s\|}{Log\, s}\right);\ \beta_X = \sup_{1<s<\infty}\left(\frac{-Log\|D_s\|}{Log\, s}\right).$$

The numbers α_X, β_X belong to the closed interval $[0,1]$ and are called the Boyd indices of X. The Boyd indices of the r.i. function space in [LT79, Definition 2.b.1] are taken to be the reciprocals of the ones we use here (i.e., $p_X = \frac{1}{\alpha_X}$ and $q_X = \frac{1}{\beta_X}$). We only need a weaker version of Boyd's interpolation theorem, [JMST79, p. 208]:

Let $X[0,1]$ be a rearrangement invariant function space, let p, q be such that $0 < \frac{1}{q} < \beta_X \leqslant \alpha_X < \frac{1}{p} < 1$, and $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A linear transformation L, which is bounded from $L^p(\Omega, \mathfrak{F}, \mathbb{P})$ to itself and from $L^q(\Omega, \mathfrak{F}, \mathbb{P})$ to itself defines a bounded operator from $X(\Omega, \mathfrak{F}, \mathbb{P})$ into itself.

The most commonly used r.i. function spaces on $[0,1]$ besides the L^p spaces, $1 \leqslant p \leqslant \infty$, are the Orlicz function spaces. We recall the definition of the Orlicz function spaces: Let $(\Omega, \mathcal{F}, \mu)$ be a separable measure space and Φ be an Orlicz function. The Orlicz function space $L^\Phi(\Omega, \mathcal{F}, \mu)$ is the space of all (equivalence classes of) $\mathcal{F}$-measurable functions f so that

$$\textstyle\int_\Omega \Phi\left(\frac{|f|}{\rho}\right) < \infty$$

for some $\rho > 0$. The norm is defined by

$$\|f\|_\Phi = \inf\{\rho > 0; \textstyle\int_\Omega \Phi\left(\frac{|f|}{\rho}\right) \leqslant 1\}$$

In addition, we require the normalization $\Phi(1) = 1$, in order that $\|\chi_{(0,1)}\|_\Phi = 1$ in $L^\Phi[0,1]$ and $L^\Phi(0,\infty)$.

It is known that the Orlicz function space $L^\Phi[0,1]$ is reflexive if and only if there exist $\lambda_0 > 0$ and $t_0 > 0$ such that for every $\lambda > \lambda_0$ there exist positive constants c_λ, C_λ such that

$$c_\lambda \Phi(t) \leqslant \Phi(\lambda t) \leqslant C_\lambda \Phi(t), \quad \forall t \geqslant t_0.$$

Moreover, the Orlicz function space $L^\Phi[0,1]$ is reflexive if and only if it is super-reflexive, (see [KP98, Corollary 2]). The Boyd indices for the Orlicz function space $L^\Phi[0,1]$ are nontrivial (i.e., $0 < \beta_\Phi \leqslant \alpha_\Phi < 1$) if and only if it is reflexive. Furthermore, we recall that any Orlicz function space $L^\Phi[0,1]$ is isomorphic to $L^2[0,1]$ if and only if there exists t_0 such that $\Phi(t)$ is equivalent to t^2 for all $t \geqslant t_0$.

Many results were proved about M-ideals and property (M) of Banach spaces, (see e.g. [Kal93] and [KW95]). We recall here the definition of the property (M) and some required results.
A Banach space X has property (M) if whenever $u, v \in X$ with $\|u\| = \|v\|$ and (x_n) is weakly null sequence in X then,

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|.$$

Proposition C.2.1. *[Kal93, Proposition 4.1]. A modular sequence space $X = \ell^{(\Phi_n)}$ can be equivalently normed to have property (M).*

For the weighted sequence space $\ell^{\overline{\Phi}}(\omega)$ where $\omega = (\omega_n)$ is a positive real sequence, if we consider $\Phi_n = \omega_n \cdot \overline{\Phi}$, then Φ_n is a sequence of Orlicz functions such that for any sequence $x = (x_n)$ we have $\|x\|_{(\Phi_n)} = \|x\|_{\overline{\Phi},\omega}$ (i.e., $\ell^{(\Phi_n)} = \ell^{\overline{\Phi}}(\omega)$). Therefore, $\ell^{\overline{\Phi}}(\omega)$ can be equivalently normed to have property (M).

Theorem C.2.2. *[Kal93, Theorem 4.3]. Let X be a separable order-continuous nonatomic Banach lattice. If X has an equivalent norm with property (M) then X is lattice-isomorphic to L^2.*

Now, we need the following results about subspaces of $X[0,1]$ that are isomorphic to $X[0,1]$.

Theorem C.2.3. *[JMST79, Theorem 9.1]. Let $X[0,1]$ be a r.i. function space such that $X[0,1]$ is q-concave for some $q < \infty$, the index $\alpha_X < 1$ and the Haar system in $X[0,1]$ is not equivalent to a sequence of disjoint function in $X[0,1]$. Then any subspace of $X[0,1]$ which is isomorphic to $X[0,1]$ contains a further subspace which is complemented in $X[0,1]$ and isomorphic to $X[0,1]$. In particular , the theorem holds for $X[0,1] = L^p[0,1]$, $1 < p < \infty$, and more generally, for every reflexive Orlicz function space $L^\Phi[0,1]$.*

The proof of the following corollary in the case of L^p, $1 < p < \infty$, is a straightforward consequence of Theorem (2.5.4) and the Pełczyński's decomposition method; it works also for the reflexive Orlicz function spaces $L^\Phi[0,1]$, since the Haar basis cannot be equivalent to the unit vector basis of a modular sequence space, unless $L^\Phi[0,1]$ is the Hilbert space.

Corollary C.2.4. *[JMST79, Corollary 9.2]. Let $X[0,1]$ be a r.i. function space satisfying the assumptions of Theorem (C.2.3). If Y is a complemented subspace of $X[0,1]$ which contains an isomorphic copy of $X[0,1]$, then $X[0,1]$ is isomorphic to Y.*

C.3 Complemented embedding of separable rearrangement invariant function spaces into spaces with unconditional Schauder decompositions

We aim to extend [BRS81, Theorem 1.1] to the r.i. function space $X[0,1]$ with the Boyd indices $0 < \beta_X \leq \alpha_X < 1$. Our proof heavily relies on the proof of [BRS81]. We will use interpolation arguments to extend it.

Theorem C.3.1. *Let $X[0,1]$ be a r.i. function space whose Boyd indices satisfy $0 < \beta_X \leq \alpha_X < 1$. Suppose $X[0,1]$ is isomorphic to a complemented subspace of a Banach space Y with an unconditional Schauder decomposition (Y_j). Then one of the following holds:*

(1) There is an i so that $X[0,1]$ is isomorphic to a complemented subspace of Y_i.

(2) A block basic sequence of the Y_i's is equivalent to the Haar basis of $X[0,1]$ and has closed linear span complemented in Y.

We first need some facts about unconditional bases and decompositions that were mentioned in [BRS81]. Given a Banach space B with unconditional basis (b_i) and (x_i) a sequence of non-zero elements in B, say that (x_i) is disjoint if there exist disjoint subsets $M_1, M_2, \cdots$ of $\mathbb{N}$ with $x_i \in [b_j]_{j\in M_i}$ for all i. Say that (x_i) is essentially disjoint if there exists a disjoint sequence (y_i) such that $\Sigma\|x_i - y_i\|/\|x_i\| < \infty$. If (x_i) is essentially disjoint, then (x_i) is essentially a block basis of a permutation of (b_i). Also, (x_i) is unconditional basic sequence. Throughout this paper, if $\{b_i\}_{i\in I}$ is an indexed family of elements of a Banach space B, $[b_i]_{i\in I}$ denotes the closed linear span of $\{b_i\}_{i\in I}$ in B. We recall the definition of the Haar system (h_n) which is normalized in L^∞: let $h_1 \equiv 1$ and for $n = 2^k + j$ with $k \geqslant 0$ and $1 \leqslant j \leqslant 2^k$,

$$h_n = \chi_{[\frac{j-1}{2^k}, \frac{2j-1}{2^{k+1}})} - \chi_{[\frac{2j-1}{2^{k+1}}, \frac{j}{2^k})}.$$

Moreover, the Haar system is unconditional basis of the r.i. function space $X[0,1]$ if and only if the Boyd indices of $X[0,1]$ satisfy $0 < \beta_X \leqslant \alpha_X < 1$, see [LT79, Theorem 2.c.6]. We use $[f = a]$ for $\{t;\ f(t) = a\}$; μ is the Lebesgue measure. For a measurable function f, *supp* $f = [f \neq 0]$.
The following three results are proved in [BRS81].

Lemma C.3.2. *Let (b_n) be an unconditional basis for the Banach space B with biorthogonal functionals (b_n^*), $T : B \mapsto B$ an operator, $\epsilon > 0$, and (b_{n_i}) a subsequence of (b_n) so that (Tb_{n_i}) is essentially disjoint and $|b_{n_i}^* Tb_{n_i}| \geqslant \epsilon$ for all i. Then (Tb_{n_i}) is equivalent to (b_{n_i}) and $[Tb_{n_i}]$ is complemented in B.*

Lemma C.3.3. *Let Z and Y be Banach spaces with unconditional Schauder decompositions (Z_i) and (Y_i) respectively; and let (P_i) (resp. Q_i) be the natural projection from Z (resp. Y) onto Z_i (resp. Y_i). Then if $T : Z \mapsto Y$ is a bounded linear operator, so is $\sum Q_i T P_i$.*

Scholium C.3.4. *Let Y be Banach space with unconditional Schauder decomposition with corresponding projections (Q_i), and let Z be a complemented subspace of Y with unconditional basis (z_i) with biorthogonal functionals (z_i^*). Suppose there exist $\epsilon > 0$, a projection $U : Y \mapsto Z$ and disjoint subsets M_1, $M_2, \cdots$ of $\mathbb{N}$ with the following properties:*
(a) $(UQ_i z_l)_{l \in M_i, i \in \mathbb{N}}$ is essentially disjoint sequence.
(b) $|z_l^*(UQ_i z_l)| \geqslant \epsilon$ for all $l \in M_i$, $i \in \mathbb{N}$.
Then $(Q_i z_l)_{l \in M_i, i \in \mathbb{N}}$ is equivalent to $(z_l)_{l \in M_i, i \in \mathbb{N}}$ and $[Q_i z_l]_{l \in M_i, i \in \mathbb{N}}$ is complemented in Y.

The next Lemma is proved in [LT79, Theorem (2.d.10)] and it is the extension of the fundamental result of Gamlen and Gaudet [GG73] to separable r.i function spaces $X[0,1]$.

Lemma C.3.5. *Let $I \subset \mathbb{N}$ such that if $E = \{t \in [0,1]; t \in supp\ h_i\ for\ infinitely\ many\ i \in I\}$, then E is of positive Lebesgue measure. Then $[h_i]_{i \in I}$ is isomorphic to $X[0,1]$.*

Recall that $X(\ell^2)$ is the completion of the space of all sequences $(f_1, f_2, \cdots)$ of elements of X which are eventually zero, with respect to the norm

$$\|(f_1, f_2, \cdots)\|_{X(\ell^2)} = \|(\sum |f_i|^2)^{\frac{1}{2}}\|_X.$$

Let $X[0,1]$ be a separable r.i function space with $0 < \beta_X \leqslant \alpha_X < 1$. Let $\{h_i\}_i$ be the Haar basis of $X[0,1]$, fix i and let (h_{ij}) be the element of $X(\ell^2)$ whose j-th coordinate equals h_i, all other coordinates 0. Then $(h_{ij})_{i,j}$ is an unconditional basis for $X(\ell^2)$, [LT79, Proposition 2.d.8]. Next, we recall [JMST79, Lemma 9.7] and for more see [LT79].

Scholium C.3.6. *There is a constant K so that for any function $j : \mathbb{N} \mapsto \mathbb{N}$, $(h_{ij(i)})_{i \geqslant 1}$ in $X(l_2)$ is $K-$ equivalent to (h_i) in X.*

The following is a consequence of the proof of [LT79, Theorem (2.d.11)] that $X[0,1]$ is primary. Let $(h_{ij}^*)_{i,j}$ denote the biorthogonal functionals to $(h_{ij})_{i,j}$.

Scholium C.3.7. *Let $T : X(l_2) \mapsto X(l_2)$ be a given operator. Suppose there is a $c > 0$ so that when $I = \{i : |h_{ij}^* T h_{ij}| \geqslant c$ for infinitely many $j\}$, then E has positive Lebesgue measure, where*

$$E = \{t \in [0,1];\ t \in supp\, h_i \text{ for infinitely many } \ i \in I\}$$

Then there is a subspace Y of $X(l_2)$ with Y isomorphic to X, $T|Y$ an isomorphism, and TY complemented in $X(l_2)$.

Proof. Fix $i \in I$, by the definition of I, there is a sequence $j_1 < j_2 < \cdots$ with $\|Th_{ij_k}\| \geqslant c > 0$ for all k. Since $\{h_{ij_k}\}_{k=1}^{\infty}$ is equivalent to the unit vectors in l_2 then it is weakly null and so $\{Th_{ij_k}\}_{k\geqslant 1}$ is weakly null. Thus, there exists $j : I \mapsto \mathbb{N}$ such that $\{Th_{ij(i)}\}_{i\in I}$ is equivalent to a block basis $(z_i)_{i\geqslant 1}$ and we can choose it such that $\sum_{i\in I} \frac{\|z_i - Th_{ij(i)}\|}{\|Th_{ij(i)}\|} < \infty$ by [BP58, Theorem (3)]. Thus $\{Th_{ij(i)}\}_{i\in I}$ is essentially disjoint with respect to $\{h_{ij}\}_{i,j=1}^{\infty}$ and $|h^*_{ij(i)}Th_{ij(i)}| \geqslant c$. Then by Lemma (C.3.2), $[Th_{ij(i)}]_{i\in I}$ is complemented in $X(l_2)$, and $(Th_{ij(i)})_{i\in I}$ is equivalent to $(h_{ij(i)})_{i\in I}$, which is equivalent to $(h_i)_{i\in I}$ by Scholium (C.3.6). In turn, $[h_i]_{i\in I}$ is isomorphic to $X[0,1]$, by Lemma (C.3.5). □

Corollary C.3.8. *Let $T : X[0,1] \mapsto X[0,1]$ be a given operator. Then for $S = T$ or $I - T$, there exists a subspace Y of $X[0,1]$ with Y isomorphic to $X[0,1]$, $S|Y$ an isomorphism, and $S(Y)$ complemented in $X[0,1]$.*

Proof. Since $X[0,1]$ is isomorphic to $X(l_2)$ by [LT79, Proposition 2.d.4], we can prove the statement with respect to $X(l_2)$. For each i, j, at least one of the numbers $|h^*_{ij}Th_{ij}|$ and $|h^*_{ij}(I-T)h_{ij}|$ is $\geqslant \frac{1}{2}$. Let $I_1 = \{i : |h^*_{ij}Th_{ij}| \geqslant \frac{1}{2}$ *for infinitely many* $j\}$, $I_2 = \{i : |h^*_{ij}(I-T)h_{ij}| \geqslant \frac{1}{2}$ *for infinitely many* $j\}$. Then $\mathbb{N} = I_1 \bigcup I_2$, hence, for $k = 1$ *or* 2, $E_k = \{t \in [0,1];\ t \in supp\, h_i$ *for infinitely many* $\ i \in I_k\}$ has positive Lebesgue measure and by Scholium (C.3.7) we get the result. □

Theorem C.3.9. *Let Z and Y be given Banach spaces. If $X[0,1]$ is isomorphic to a complemented subspace of $Z \oplus Y$, then $X[0,1]$ is isomorphic to a complemented subspace of Z or to a complemented subspace of Y.*

Proof. Let P (resp. Q) denote the natural projection from $Z \bigoplus Y$ onto Z (resp. Y). Hence $P + Q = I$. Let K be a complemented subspace of $Z \bigoplus Y$ isomorphic to $X[0,1]$ and let $U : Z \bigoplus Y \mapsto K$ be a projection. Since $UP|K + UQ|K = I|K$, Corollary (C.3.8) shows that there is a subspace W of K with W isomorphic to $X[0,1]$, $T|W$ an isomorphism, and TW complemented in K, where $T = UP|K$ or $T = UQ|K$. Suppose for instance the former: Let S be a projection from K onto TW and $R = (T|W)^{-1}$. Then $I|W = RSUP|W$; hence since the identity on W may be factored through Z, W is isomorphic to a complemented subspace of Z. □

The next lemma is [GG73, Lemma 4] and it is pointed out in the proof of [LT79, Theorem (2.d.10)] for separable r.i function space.

Lemma C.3.10. *Let (x_i) be a measurable functions on $[0,1]$ with x_1 $\{0,1\}-$valued and x_i $\{1,0,-1\}-$ valued for $i > 1$. Suppose there exist positive constants a and b so that, for all positive l, with k the unique integer, $1 \leqslant k \leqslant l$, and α the unique choice of $+1$ or -1 so that supp $h_{l+1} = [h_k = \alpha]$, then*

(a) $[x_k = \alpha] =$ supp x_{l+1}

(b) $\frac{a}{2}\int |h_k| \leqslant \mu([x_{l+1} = \beta]) \leqslant \frac{b}{2}\int |h_k|$ for $\beta = \pm 1$

Then (x_n) is equivalent to (h_n) in $X[0,1]$, and $[x_n]$ is the range of a one-norm projection defined on $X[0,1]$.

Proof. Let $A_n = supp\ x_n$, then the subspace $[x_n]$ is complemented in $X[0,1]$ since it is the range of the projection $P(f) = \chi_A \mathbb{E}_{\mathcal{F}}(f)$ of norm one, where $A = \bigcup_{n=1}^{\infty} A_n$ and $\mathbb{E}_{\mathcal{F}}$ denotes the conditional expectation operator with respect to the sub-σ-algebra generated by the measurable sets A_n.
Since (x_n) and (h_n) are equivalent in $L^p[0,1]$ for all $1 < p < \infty$, by [GG73, Lemma 4], then the operator $R_1 : L^p[0,1] \mapsto L^p[0,1]$ defined by $R_1(\sum_{n=1}^{\infty} a_n h_n) = \sum_{n=1}^{\infty} a_n x_n$ is bounded for all $1 < p < \infty$. Therefore, the Boyd interpolation theorem implies that R_1 will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leqslant \alpha_X < 1$. Now, we will define a bounded operator R_2 on $L^p[0,1]$ as follows: if $P(f) = \sum_{n=1}^{\infty} a_n x_n$, then $R_2(f) = \sum_{n=1}^{\infty} a_n h_n$. Again, the Boyd interpolation theorem implies that R_2 will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leqslant \alpha_X < 1$, and this clearly yields the equivalence of the sequences (x_n) and (h_n) in $X[0,1]$. □

Scholium C.3.11. *Let (z_i) be a sequence of measurable functions on $[0,1]$ such that z_1 is $\{0,1\}$-valued nonzero in L^1 and z_i is $\{0,-1,1\}$- valued with $\int z_i = 0$ for all $i > 1$. Suppose that for all positive l, letting k be the unique integer, $1 \leqslant k \leqslant l$, and α the unique choice of 1 or -1 so that $supp\, h_{l+1} = \{t; h_k(t) = \alpha\}$, then*

$$supp\, z_{l+1} \subset \{t; z_k(t) = \alpha\}$$

and $\mu(\{t; z_k(t) = \alpha\} \sim supp\ z_{l+1}) \leqslant \epsilon_l \int |z_1|$, where $\epsilon_l = \frac{1}{2^{j^2}}$ and " $\sim$ " denotes the set difference. Then (z_n) is equivalent to (h_n), and $[z_n]$ is complemented in X.

Proof. The proof is depending on [BRS81, Scholium (1.11)] and The Boyd interpolation theorem with same procedure in the proof of Lemma (C.3.10). □

Now, We will use the same procedure of [BRS81, Theorem 1.1] in order to prove Theorem (C.3.1).
We assume that $X(\ell^2)$ is a complemented subspace of Y; let $U : Y \mapsto X(\ell^2)$ be a projection. Let $(Y_i)_i$ be an unconditional decomposition of Y. Suppose that (1) fails, that is, there is no i with $X[0,1]$ isomorphic to a complemented subspace of Y_i. We shall then construct a blocking of the decomposition (Y_i) with corresponding projections (Q_i), finite disjoint subsets $M_1,\ M_2, \cdots$ of $\mathbb{N}$, and a map $j : \cup_{i=1}^{\infty} M_i \mapsto \mathbb{N}$ so that:

(i) $(Q_k h_{ij(i)})_{i\in M_k,\ k\in\mathbb{N}}$ is equivalent to $(h_i)_{i\in M_k,\ k\in\mathbb{N}}$ with $[Q_k h_{ij(i)}]_{i\in M_k,\ k\in\mathbb{N}}$ complemented in $X(\ell^2)$.

(ii) (z_k) is equivalent to the Haar basis and $[z_k]$ is complemented in $X[0,1]$, where $z_k = \sum_{i\in M_k} h_i$ for all k.

We simply let $b_k = \sum_{i\in M_k} Q_k h_{ij(i)}$ for all k; then (b_k) is the desired block basic sequence equivalent to the Haar basis with $[b_k]$ complemented.
Let P_i be the natural projection from Y onto Y_i. More generally, for F a subset of $\mathbb{N}$ we let $P_F = \sum_{i\in F} P_i$. Also, we let $R_n = I - \sum_{i=1}^{n} P_i\ (= P_{(n,\infty)})$. We first draw a consequence from our assumption that no Y_i contains a complemented isomorphic copy of $X[0,1]$.

Lemma C.3.12. *For each n, let*

$$I = \{i \in \mathbb{N}; h_{ij}^* U R_n h_{ij} > \frac{1}{2} \text{ for infinitely many integers } j\}. \tag{C.3.1}$$

Let $E_I = \{t \in [0,1]; \text{t belongs to the support of } h_i \text{ for infinitely many integers } i \in I\}$. Then $\mu(E_I) = 1$ (where μ is the Lebesgue measure).

Indeed, let $L = \{i \in \mathbb{N}; h_{ij}^* U P_{[1,n]} h_{ij} \geqslant \frac{1}{2}$ *for infinitely many integers* $j\}$; then $I \cup L = \mathbb{N}$.

Hence $E_I \bigcup E_L = [0,1]$. So if $\mu(E_I) < 1, \mu(E_L) > 0$. But then $T = UP_{[1,n]}$ satisfies the hypotheses of Scholium (C.3.7). Hence there is a subspace Z of $X(\ell^2)$, with Z isomorphic to $X[0,1]$ and TZ complemented in $X(\ell^2)$. It follows easily that $P_{[1,n]}|Z$ is an isomorphism with $P_{[1,n]}Z$ complemented; that is, X[0,1] embeds as a complemented subspace of $Y_1 \bigoplus \cdots \bigoplus Y_n$. Hence by Theorem (C.3.9), $X[0,1]$ embeds as a complemented subspace of Y_i for some i, a contradiction.

Lemma C.3.13. *Let $I \subset \mathbb{N}$, E_I as in Lemma (C.3.12) with $\mu(E_I) = 1$, and $S \subset [0,1]$ with S a finite union of disjoint left-closed dyadic intervals. Then there exists a $J \subset I$ so that $supp\, h_i \bigcap supp\, h_l = \emptyset$, for all $i \neq l$, i, $l \in J$, with $S \supset \cup_{j\in J}$ $supp\, h_j$ and $S \sim \cup_{j\in J}$ $supp\, h_j$ of measure zero.*

Proof. It suffices to prove the result for S equal to the left-closed dyadic interval. Now any two Haar functions either have disjoint supports, or the support of one is contained in that of the other. Moreover, for all but finitely many $i \in I$, $supp\ h_i \subset S$ or $supp\, h_i \cap S = \emptyset$. Hence S differs from $\bigcup\{supp\, h_j :\ supp\, h_i \subset S,\ j \in I\}$ by a measure-zero set. Now simply let $J = \{j \in I;\ supp\, h_j \subset S,\ and\ there\ is\ no\ l \in I\ with\ supp\, h_j \subset supp\, h_l \subset S\}$. □

We now choose M_1, $M_2, \cdots$ disjoint finite subsets of $\mathbb{N}$, a map $j : \cup_{i=1}^{\infty} M_i \mapsto \mathbb{N}$, and $1 = m_0 < m_1 < m_2 < \cdots$ with the following properties:

A. For each k, the h_i*'s* for $i \in M_k$ are disjointly supported. Set $z_k = \sum_{i\in M_k} h_i$. Then z_k satisfies the hypotheses of Scholium (C.3.11)

B. Let $Q_k = P_{[m_{k-1},m_k]}$ for all k. Then $(UQ_k h_{ij(i)})_{i\in M_k, k\in\mathbb{N}}$ is essentially disjoint and $h^*_{ij(i)} UQ_k h_{ij(i)} > \frac{1}{2}$ for all $i \in M_k,\ k \in \mathbb{N}$.

Having accomplished this, we set $b_k = \sum_{i\in M_k} Q_k h_{ij(i)}$ for all k. Then by B, (b_k) is a block basic sequence of the Y_i*'s*. By Scholium (C.3.4),

$$(Q_k h_{ij(i)})_{i\in M_k,\ k\in\mathbb{N}} \approx (h_{ij(i)})_{i\in M_k,\ k\in\mathbb{N}} \approx (h_i)_{i\in M_k,\ k\in\mathbb{N}} \tag{C.3.2}$$

where " $\approx$ " denotes equivalence of basic sequences; the last equivalence follows from Scholium (C.3.6) i.e., the unconditionality of the Haar basis. Hence by the definitions of (b_k) and (z_k), (b_k) is equivalent to (z_k) which is equivalent to (h_k), the Haar basis, by Scholium (C.3.11). Also, since $[z_k]$ is complemented in $X[0,1]$ by Scholium (C.3.11), $[b_k]$ is complemented in $[Q_k h_{ij(i)}]_{i\in M_k,\ k\in\mathbb{N}}$ by (C.3.2). Again by Scholium (C.3.4), $[Q_k h_{ij(i)}]_{i\in M_k,\ k\in\mathbb{N}}$ is complemented in Y, hence also $[b_k]$ is complemented in Y.

It remains now to choose the M_i*'s*, m_i*'s* and the map j. To insure B, we shall also choose a sequence $(f_i)_{i\in M_k,\ k\in\mathbb{N}}$ of disjointly supported elements of $X(\ell^2)$ (disjointly supported with respect to the basis (h_{ij})) so that

$$\sum_{i\in M_k} \frac{\|UQ_k h_{ij(i)} - f_i\|}{\|UQ_k h_{ij(i)}\|} < \frac{1}{2^k},\ for\ all\ k \tag{C.3.3}$$

To start, we let $M_1 = \{1\}$ and $j(1) = 1$. Thus $z_1 = 1$; we also set $f_1 = h_{11}$. Then $h_{11} = Uh_{11} = lim_{n\to\infty} UP_{[1,n]} h_{11}$. So it is obvious that we can choose $m_1 > 1$ such that $\|UP_{[1,m_1]} h_{11} - h_{11}\| < \frac{1}{2}$; hence $h^*_{11} UP_{[1,m_1]} h_{11} > \frac{1}{2}$. Thus, the first step is essentially trivial.

Now suppose $l \geqslant 1$, $M_1, \cdots, M_l$, $m_1 < \cdots < m_l$, $j : \cup_{i=1}^{l} M_i \mapsto \mathbb{N}$ and $(f_i)_{i\in M_k,\ 1<k<l}$ have been chosen.We set $z_i = \sum_{j\in M_i} h_j$ for all i, $1 \leqslant i \leqslant l$.

Let $1 \leqslant k \leqslant l$ be the unique integer and α the unique choice of ± 1 so that $supp\ h_{l+1} = [h_k = \alpha]$. Let $S = [z_k = \alpha]$. Set $n = m_l$ and let I be as in Lemma (C.3.12). Since S is a finite union of disjoint left-closed dyadic intervals, by Lemma (C.3.13) we may choose a finite set $M_{l+1} \subset I$, disjoint from $\cup_{i=1}^{l} M_i$, so that the h_i*'s* for $i \in M_{l+1}$ are disjointly supported with *supp* $h_i \subset S$ for $i \in M_{l+1}$

$$\mu(S \sim \cup_{i \in M_{l+1}} supp\ h_i) \leqslant \epsilon_l \tag{C.3.4}$$

(where $\epsilon_j = \frac{1}{2^{j^2}}$ for all j). At this point, we have that $z_{l+1} = \sum_{i \in M_{l+1}} h_i$ satisfies the conditions of Scholium (C.3.11).
By the definition of I, for each $i \in M_{l+1}$ there is an infinite set J_i with

$$h_{ij}^* U R_n h_{ij} > \frac{1}{2},\ for\ all\ j \in J_i.$$

Now $(UR_n h_{ij})_{j=1}^{\infty}$ is a weakly null sequence; hence it follows that we may choose $j : M_{l+1} \mapsto \mathbb{N}$ and disjointly finitely supported elements $(f_i)_{i \in M_{l+1}}$, with supports (relative to the h_{ij}*'s*) disjoint from those of $\{f_i : i \in \cup_{i=1}^{l} M_i\}$, so that

$$\sum_{i \in M_{l+1}} \frac{\|UR_n h_{ij(i)} - f_i\|}{\|UR_n h_{ij(i)}\|} < \frac{1}{2^{l+1}}$$

At last, since $R_n g = lim_{k \to \infty} P_{[m_l,k)} g$ for any $g \in X(\ell^2)$, we may choose an $m_{l+1} > m_l$ so that setting $Q_{l+1} = P_{[m_l, m_{l+1})}$, (C.3.3) holds for $k = l+1$ and also

$$h_{ij}^* U Q_k h_{ij(i)} > \frac{1}{2},\ for\ all\ i \in M_k$$

This completes the construction of the M_i*'s*, m_i*'s* and map j. Since (C.3.2) holds, A and B hold. Thus (2) of Theorem (C.3.1) holds; thus the proof is complete.

C.4 Main results

In this section we will extend [Bou81, Theorem (4.30)] to the Orlicz function spaces. We will use in particular the Boyd interpolation theorem and Kalton's result [Kal93], see Theorem (C.2.2)
Again we let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$. The space $X(G)$ is a r.i. function space defined on the separable measure space consisting of the Cantor group $G = \{-1,1\}^{\mathcal{C}}$ equipped with the Haar measure. The Walsh functions w_F where F is a finite subset of $\mathcal{C}$ generate the $L^p(G)$ spaces for all $1 \leq p < \infty$. Then they also generate the r.i. function space $X(G)$.
We consider a r.i. function space $X[0,1]$ such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. The subspace $X_{\mathcal{C}}$ is the closed linear span in the r.i. function space $X(G)$ over all finite branches Γ in $\mathcal{C}$ of the functions which depend only on the Γ-coordinates. Thus $X_{\mathcal{C}}$ is a subspace of $X(G)$ generated by Walsh functions $\{w_\Gamma = \prod_{c \in \Gamma} r_c;\ \Gamma$ *is a finite branch of* $\mathcal{C}\}$.

Proposition C.4.1. *Let $X[0,1]$ be a r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Then $X_{\mathcal{C}}$ is a complemented subspace in $X(G)$.*

Proof. The authors in [Bou81] and [BRS81] express the orthogonal projection P on $X_{\mathcal{C}}^p$ which is bounded in L^p-norm for all $1 < p < \infty$, by taking $\beta_\emptyset$ = trivial algebra and $\beta_c = \mathfrak{G}(d \in \mathcal{C}; d \leqslant c)$ for each $c \in \mathcal{C}$. For $c \in \mathcal{C}$ and $|c| = 1$, let $c' = \emptyset$ and for $c \in \mathcal{C}$ such that $|c| > 1$, let c' be the predecessor of c in $\mathcal{C}$.
The orthogonal projection P is given by

$$P(f) = \mathbb{E}[f|\beta_\emptyset] + \sum_{c \in \mathcal{C}} (\mathbb{E}[f|\beta_c] - \mathbb{E}[f|\beta_{c'}]) \tag{C.4.1}$$

for every $f \in L^p(G)$, $1 < p < \infty$.
Let the Boyd indices of $X[0,1]$ satisfy $0 < \beta_X \leqslant \alpha_X < 1$. Then the Boyd interpolation theorem implies that the map P is a bounded projection on $X_{\mathcal{C}}$ for all r.i. function spaces $X(G)$. Therefore, $X_{\mathcal{C}}$ is complemented subspace of $X(G)$. □

Since the elements of any finite subset of the infinite branch Γ_∞ are mutually comparable, then it is a branch. Thus, the subspace X_{Γ_∞} is isometrically isomorphic to $X(\{-1,1\}^{\Gamma_\infty})$ and a one-complemented subspace of $X(G)$ by the conditional expectation operator.

For a tree T of $\mathcal{C}$, we define the subspace X_T of $X(G)$ as a closed linear span in $X(G)$ over all finite branches Γ in T of all those functions in $X(G)$ which depend only on the coordinates of Γ.
By using the conditional expectation with respect to the sub-σ-algebra generated by a tree T of $\mathcal{C}$, one can find that X_T is a one-complemented subspace of $X_{\mathcal{C}}$ and so it is a complemented subspace of $X(G)$. Therefore, the next result is true.

Theorem C.4.2. *Let $X[0,1]$ be a r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leqslant \alpha_X < 1$, and T be a tree on $\mathbb{N}$. Then X_T is a complemented subspace of $X(G)$.*

The next proposition is a direct consequence of Corollary (C.2.4).

Proposition C.4.3. *Let $X[0,1]$ be a r.i. function space such that $X[0,1]$ is q-concave for some $q < \infty$, the index $\alpha_X < 1$ and the Haar system in $X[0,1]$ is not equivalent to a sequence of disjoint function in $X[0,1]$. Then $X_{\mathcal{C}}$ is isomorphic to $X(G)$.*

Theorem C.4.4. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $L^\Phi[0,1]$ does not embed in X_T^Φ if and only if T is a well founded tree.*

Proof. If T contains an infinite branch, then obviously $L^\Phi[0,1]$ embeds in X_T^Φ.
We want to show that if T is well founded, then $L^\Phi[0,1]$ does not embed in X_T^Φ. We will use Theorem (C.2.3) and Theorem (C.3.1):
We proceed by induction on $\circ[T]$. Assume the conclusion fails. Let T be a well founded tree such that $\circ[T] = \alpha$ and $L^\Phi[0,1]$ embeds in X_T^Φ, where $\alpha = \min\{\circ[T];\ L^\Phi[0,1] \textit{ embeds in } X_T^\Phi\}$.
We write $T = \bigcup_n (n, T_n)$, with $\circ[T] = \sup\limits_n(\circ[T_n] + 1)$.

The space X_T^Φ is generated by the sequence of probabilistically mutually independent spaces $B_n = X^\Phi_{(n,T_n)}$. In particular, $\oplus_n B_n$ is an unconditional decomposition of X_T^Φ by the inequality in [BG70, Corollary(5.4)] that is: let $x_1, x_2, \cdots$ be an independent sequence of random variables, each with expectation zero, then for every $n \geqslant 1$

$$c \int_\Omega \Phi\left([\sum_{k=1}^n x_k^2]^{\frac{1}{2}}\right) \leqslant \int_\Omega \Phi\left(|\sum_{k=1}^n x_k|\right) \leqslant C \int_\Omega \Phi\left([\sum_{k=1}^n x_k^2]^{\frac{1}{2}}\right) \tag{C.4.2}$$

By Theorem (C.2.3), the space $L^\Phi[0,1]$ embeds complementably in X_T^Φ. Application of Theorem (C.3.1) implies that **A** or **B** below is true:

A. There is some n such that $L^\Phi[0,1]$ is isomorphic to a complemented subspace of B_n.

B. there is a block basic sequence (b_r) of the B_n*'s* which is equivalent to the Haar system of $L^\Phi[0,1]$.

Assume (A): It is easily seen that B_n is isomorphic to $X_{T_n}^\Phi \bigoplus X_{T_n}^\Phi$. So by another application of Theorem (C.3.1), $L^\Phi[0,1]$ should embed complementably in $X_{T_n}^\Phi$. This however is impossible by induction hypothesis since $\circ[T_n] < \circ[T]$.

Assume (B): A block basic sequence of the B_n's is a sequence of probabilistically independent functions which is equivalent to the Haar system of $L^{\Phi}[0,1]$. By [Ros70, Definition 3], we have that $L^{\Phi}[0,1]$ is isomorphic to a weighted Orlicz sequence space $\ell^{\overline{\Phi}}(\omega)$ where $\overline{\Phi}$ is an Orlicz function which is equivalent to Φ at ∞ and with t^2 if $t \in [0,1]$, moreover, $\sum_{n=1}^{\infty} \omega_n = \infty$ and $\omega_n \to 0$ as $n \to \infty$. However, Proposition (C.2.1) and Theorem (C.2.2) imply that this space does not contain an $L^{\Phi}[0,1]$ copy. This contradiction concludes the proof. $\square$

Let $\mathcal{T}$ be the set of all trees on $\mathbb{N}$ which is a closed subset of the Cantor space $\Delta = 2^{\mathbb{N}^{<\mathbb{N}}}$. In addition, we denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure for more about the application of descriptive set theory in the geometry of Banach spaces see e.g., [Bos02], or [AGR03].

Lemma C.4.5. *Suppose $\psi : \mathcal{T} \mapsto \mathcal{SE}$ is a Borel map, such that if T is well founded tree and S is a tree and $\psi(T) \cong \psi(S)$, then the tree S is well founded. Then there are uncountably many mutually non-isomorphic members in the class $\{\psi(T); T \text{ is well founded tree}\}$.*

Proof. Assume by contradiction that the number of the non-isomorphic members in the class

$$\{\psi(T);\ T \text{ is well founded tree}\}$$

is countable, then there exists a countable sequence of well founded trees $(T_i)_{i=1}^{\infty}$ such that for any well founded tree T there exists i such that $\psi(T)$ is isomorphic to $\psi(T_i)$.
Consider $B_i = \{X \in \mathcal{SE};\ X \cong \psi(T_i)\}$, then B_i is an analytic subset of $\mathcal{SE}$ for all $i \geqslant 1$ because of the analyticity of the isomorphism relation. Moreover, since ψ is Borel map, then $A_i = \{T \in \mathcal{T};\ \psi(T) \cong \psi(T_i)\}$ is analytic subset of $\mathcal{T}$ for all $i \geqslant 1$.
From our hypothesis we get that $\{T;\ T \text{ is well founded tree}\} = \cup_{i \geqslant 1} A_i$ is analytic which is a contradiction. $\square$

Lemma C.4.6. *The map $\psi : \mathcal{T} \mapsto \mathcal{SE}$ defined by $\psi(T) = X_T^{\Phi}$ is Borel.*

Proof. Let U be an open set of $C(\Delta)$ and $(\Gamma_i)_{i=1}^{\infty}$ be a sequence of all the finite branches of $\mathcal{C}$. It is sufficient to prove that $\mathcal{B} = \{T \in \mathcal{T}; \psi(T) \cap U \neq \emptyset\}$ is Borel. It is clear that $\psi(T) \cap U \neq \emptyset$ if and only if there exists $\underline{\lambda} = (\lambda_i)_{i=1}^{n} \in \mathbb{Q}^{<\mathbb{N}}$ such that $\sum_{i=0}^{n} \lambda_i w_{\Gamma_i} \in U$ and $\lambda_i = 0$ when $\Gamma_i \not\subset T$.
Let $\Lambda = \{\underline{\lambda} \in \mathbb{Q}^{<\mathbb{N}}; \sum_i \lambda_i w_{\Gamma_i} \in U\}$ and for $\underline{\lambda} \in \mathbb{Q}^{<\mathbb{N}}$ set $\text{supp}(\underline{\lambda}) = \{i \in \mathbb{N}; \lambda_i \neq 0\}$. Then

$$\mathcal{B} = \bigcup_{\underline{\lambda} \in \Lambda} \bigcap_{i \in supp(\underline{\lambda})} \{T \in \mathcal{T}; \Gamma_i \subset T\}. \tag{C.4.3}$$

Since $\{T \in \mathcal{T}; \Gamma_i \subset T\} = \bigcap_{c \in \Gamma_i} \{T \in \mathcal{T}; c \in T\}$ is Borel (because for $c \in \mathcal{C}$ the set $\{T \in \mathcal{T}; c \in T\}$ is clopen subset in $\mathcal{T}$), then $\mathcal{B}$ is Borel. $\square$

We recall that in [JMST79, p. 235], it is shown that the Orlicz function $\Phi(t) = t^2 \exp(f_0(\log(t)))$, where $f_0(u) = \sum_{k=1}^{\infty}(1 - \cos \frac{\pi u}{2^k})$ is such that the associated Orlicz function space $L^{\Phi}[0,1]$ and $L^{\Phi}[0,\infty)$ are isomorphic. Moreover, this space is $(2-\epsilon)$-convex and $(2+\epsilon)$-concave for all $\epsilon > 0$. Hence, the Boyd indices satisfy $\alpha_\Phi = \beta_\Phi = \frac{1}{2}$. In addition, the space $L^{\Phi}[0,1]$ is not isomorphic to $L^2[0,1]$. Also, Orlicz function spaces are constructed in [HP86] and [HR89] which do not contain any complemented copy of ℓ^p for $p \geqslant 1$. Thus, the next corollary is not a straightforward consequence of [BRS81].

Corollary C.4.7. *Let $L^{\Phi}[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^{\Phi}[0,1]$.*

Proof. Let ψ be the Borel map defined by $\psi(T) = X_T^\Phi$. Now, let T be a well founded tree and T_0 be a tree such that $X_{T_0}^\Phi$ is isomorphic to a subspace of X_T^Φ, then Theorem (C.4.4) implies that T_0 is well founded. By Theorem (C.4.2), the spaces X_T^Φ are complemented in $L^\Phi(G)$. Hence, there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0,1]$ by Lemma (C.4.5). $\square$

It was mentioned before that the set of all well founded trees is co-analytic non Borel and so the set of all trees which are not well founded (ill founded) is analytic non-Borel. Following [Bos02], if X is a separable Banach space, then $< X >$ denotes the equivalence class $\{Y \in \mathcal{SE};\ Y \simeq X\}$. We have the following result.

Corollary C.4.8. *Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $< L^\Phi[0,1] >$ is analytic non Borel.*

Proof. Since the isomorphism relation $\{(X,Y);\ X \simeq Y\}$ is analytic in $\mathcal{SE}^2$ by [Bos02, Theorem 2.3], then the class $< L^\Phi[0,1] >$ is analytic. Moreover, since ψ is Borel and $\psi^{-1}(< L^\Phi[0,1] >) = \{T;\ T \textit{ is ill founded}\}$ by Theorem (C.4.4), then the class $< L^\Phi[0,1] >$ is non-Borel. $\square$

In [Bos02], it has been shown that $< \ell^2 >$ is Borel. It is unknown whether this condition characterizes the Hilbert space, and thus we recall [Bos02, Problem 2.9]: Let X be a separable Banach space whose isomorphism class $< X >$ is Borel. Is X isomorphic to ℓ^2? A special case of this problem seems to be of particular importance, namely: Is the isomorphism class $< c_0 >$ of c_0 Borel? For more about this question and analytic sets of Banach spaces see [God10].

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